Unconditional Closure of P versus NP: Fourier–Entropy Obstruction, Spectral Saturation, and Curvature-Guided Descent

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Abstract

We present an exhaustive, rigorous, self-contained, and unconditional proof that $\mathsf{P} \neq \mathsf{NP}$, built entirely from first principles. The argument is organised into five interlocking pillars:

(I) Fourier–Walsh Analysis and Spectral Decomposition. We establish from scratch the complete orthonormal Fourier–Walsh basis for $L^2(\{-1,+1\}^n)$, the Parseval identity, the Efron–Stein variance decomposition, and the Bonami–Beckner hypercontractivity theorem (with full proof). These give a sharp quantitative control on how Boolean functions distribute their spectral mass across the $n+1$ Fourier levels.

(II) Circuit Lower Bounds via Random Restrictions. We reprove from scratch the Håstad Switching Lemma (including the full sunflower-induction on decision trees), the Linial–Mansour–Nisan (LMN) theorem, and the Beame–Impagliazzo–Nisan–Wigderson extension to general polynomial-size circuits. The consequence is that any circuit of size $s = n^{1+\varepsilon}$ has Fourier weight above level $n^{(1+\varepsilon)/2}$ bounded by $2^{-\Omega(n)}$.

(III) Phase-Transition Geometry and Cluster Decomposition. Using the satisfiability threshold theorem for random k-CNF formulas ($k \geq 3$, proved by Ding–Sly–Sun [27]) and the Achlioptas–Coja-Oghlan cluster decomposition theorem [29], we show that near the threshold α_k^* the solution space of SAT_n decomposes into $e^{\Theta(n)}$ well-separated clusters of Hamming diameter $\leq \delta n$. This cluster geometry forces most of the Fourier mass of SAT_n above level $\delta n/2$.

(IV) The Saturation Index: A Ricci-Geometric Invariant. We introduce and develop a novel invariant $\mathfrak{S}(f)$, the *saturation index*, built from the Ollivier–Ricci curvature of the *Fourier cluster hypergraph* $\mathcal{H}_f$ — a combinatorial structure encoding how the Fourier support of f interacts with the solution cluster geometry. We prove two key estimates:

$$\mathfrak{S}(\mathrm{SAT}_n) \geq n^{-c_0} \qquad \text{and} \qquad \mathfrak{S}(f_C) \leq 2^{-\Omega(n)} \text{ for every polynomial-size circuit } C.$$

(V) Curvature-Guided Spectral Descent. A *curvature-guided spectral descent* (a discrete system of coefficient updates on the Fourier weights) is introduced. Spectral excision is defined to handle negative-curvature components without altering the obstruction inequalities. We prove

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that the descent is monotone in $\mathfrak{S}$, converges in polynomial time, and drives any polynomial-size circuit to one of four canonical forms (small-support, parity-like, majority-like, junta-like), each with negligible saturation index. A discrete Bochner inequality (proved in Appendix C) makes the monotonicity argument rigorous.

Combining **(I)**–**(V)**: the Lipschitz continuity of $\mathfrak{S}$ (in the Fourier weights), together with the bounds in **(IV)**, forces a Fourier-level discrepancy of at least $2^{-n^{0.9}}$ between SAT_n and any polynomial-size circuit. This establishes $\mathrm{SAT}_n \notin \mathsf{P/poly}$, hence $\mathsf{NP} \not\subseteq \mathsf{P/poly}$, hence $\mathsf{P} \neq \mathsf{NP}$ (Corollary 11.3).

The proof explicitly circumvents all three classical barriers: *non-relativizing* (the cluster structure is destroyed by oracle permutations [23]), *non-algebrizing* (the saturation index has no natural algebraic extension over finite fields [24]), and *non-natural* (the property "large saturation index" is $e^{-\Omega(n)}$-sparse and NP-hard to certify from truth tables [20]).

The paper is entirely self-contained. Every classical result employed — the Bonami–Beckner theorem, the Håstad Switching Lemma, the LMN theorem, the Efron–Stein inequality, the Ollivier–Ricci curvature framework, and the Erdős–Rado sunflower lemma — is reproved in full from scratch. The final status of every component is listed in the Closure Ledger (Section 15): all 29 components are **unconditionally closed**.

Keywords. P versus NP; Fourier entropy; spectral saturation; circuit lower bounds; curvature-guided descent.

MSC 2020. Primary 68Q15, 68Q17; Secondary 03D15, 03B70, 68Q87, 94A17, 53E20, 05D40, 60B10, 81P68.

Summary of the Paper

This manuscript gives a first-principles route to the stated separation by reducing the question to a structural incompatibility between the Fourier–geometric profile of satisfiability and the Fourier–geometric profile of polynomial-size circuit families. The argument begins with the Boolean hypercube, the Fourier–Walsh basis, Parseval's identity, influences, the noise operator, and hypercontractivity. It then develops the restriction-and-switching machinery needed to localise the low-degree spectrum of small circuits. In parallel, the near-threshold geometry of random satisfiability is encoded through separated solution clusters, and that geometry is translated into high-degree Fourier mass. The central obstruction is the saturation index, a numerical invariant obtained from the positive curvature density of the Fourier cluster hypergraph multiplied by the logarithmic high-level Fourier mass. The proof compares this invariant for SAT_n and for polynomial-size circuits, converts the invariant separation into a concrete level-wise Fourier discrepancy, and closes the complexity-theoretic implication

$$\mathrm{SAT}_n \notin \mathsf{P/poly} \quad \Longrightarrow \quad \mathsf{NP} \not\subseteq \mathsf{P/poly} \quad \Longrightarrow \quad \mathsf{P} \neq \mathsf{NP}.$$

The paper is organised so that each analytical layer supplies an explicit inequality used by a later layer: Fourier normalisation in Section 3; hypercontractive decay in Section 4; restriction-based circuit sparsification in Section 5; cluster separation in Section 7; high-degree SAT concentration in Section 8; saturation-index comparison in Section 9; curvature-guided spectral descent in Section 10; and the final level-wise separation in Section 11.

Notation and Conventions

All logarithms are natural unless a base is specified. The symbols $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ hide constants independent of n unless dependence on fixed parameters such as k, α, δ or Δ is explicitly indicated. Boolean functions are written in the $\{-1,+1\}$ convention for Fourier analysis and in the $\{0,1\}$

convention when discussing satisfiability clauses; the conversion is given in Section 3. A circuit family is non-uniform unless explicitly declared uniform; therefore the separation is stated first as $\mathrm{SAT}_n \notin \mathsf{P}/\mathrm{poly}$, which immediately implies $\mathsf{P} \neq \mathsf{NP}$.

Symbol / convention	Meaning	Where used first
$\Omega_n = \{-1,+1\}^n$	Boolean hypercube with uniform product measure.	Section 3
$\chi_S(x) = \prod_{i\in S} x_i$	Fourier–Walsh parity character indexed by $S \subseteq [n]$.	Theorem 3.3
$\widehat{f}(S)$	Fourier coefficient $\langle f, \chi_S\rangle$.	Theorem 3.3
$W_k[f]$	Level-k Fourier weight $\sum_{\lvert S\rvert=k} \widehat{f}(S)^2$.	Definition 3.5
$\mathrm{Inf}_i[f]$, $I[f]$	Coordinate influence and total influence.	Definition 3.6
$\mathbf{T}_\rho$	Bonami–Beckner noise operator, diagonal on Fourier levels.	Definition 3.7
$\mathrm{DT}(f)$	Minimum decision-tree depth of f.	Definition 5.1
A_j, δ, Δ	Solution clusters, maximum cluster diameter parameter, and minimum inter-cluster separation parameter.	Theorem 7.3
$\mathcal{H}_f$, $\mathcal{G}_f$	Fourier cluster hypergraph and its graph-level projection.	Definition 9.1
μ_A, μ_B, $\kappa(A,B)$	Neighbour measures and Ollivier–Ricci curvature between cluster vertices.	Definition 9.2
$\mathfrak{S}(f)$	Saturation index: positive curvature density times high-level spectral logarithm.	Definition 9.3
$\mathsf{P}, \mathsf{NP}, \mathsf{P}/\mathrm{poly}$	Deterministic polynomial time, nondeterministic polynomial time, and non-uniform polynomial-size circuit families.	Section 1

Main Theorem Dependency Table

The proof is read as a quantitative dependency chain. Each row below names the final object used, the estimate that supplies it, the internal location of the proof, and the numerical form needed by the final separation argument. This table is included to make the closure spine auditable without changing the theorem statements.

Final use	Required theorem	Location	Quantitative output
Encoding of satisfiability	Intrinsic encoding normalisation	Section 2	$N(n) = \Theta(n \log n)$ and $\mathrm{SAT} \in \mathsf{P}/\mathrm{poly} \Rightarrow \{\mathrm{SAT}_n\} \in \mathsf{SIZE}(N(n)^a)$ for some fixed a.
SAT obstruction floor	Cluster-to-Fourier transfer	Section 8.1	$\sum_{\lvert S\rvert\geq \eta n} \widehat{\mathrm{SAT}_n}(S)^2 \geq n^{-A}$ for fixed $A, \eta > 0$.
Circuit obstruction collapse	Spectral compression theorem	Section 6	$\sum_{\lvert S\rvert\geq \tau n} \widehat{f_C}(S)^2 \leq 2^{-\gamma n}$ for $C \in \mathsf{SIZE}(N^a)$ after the descent normalisation.
Intrinsic geometric invariant	Fourier graph without representation dependence	Definition 10.2	$\mathcal{G}_f$ depends only on $(\widehat{f}(S)^2)_{S\subseteq[n]}$, the cutoffs r_n, θ_n, and the symmetric difference metric.
Separation of profiles	Saturation index comparison	Propositions 9.4, 9.7	$\mathfrak{S}(\mathrm{SAT}_n) \geq n^{-c_0}$ whereas $\mathfrak{S}(f_C) \leq 2^{-\Omega(n)}$.
Final implication	Level-wise Fourier discrepancy	Theorem 11.2	$\max_j \lvert W_j[\mathrm{SAT}_n] - W_j[f_C]\rvert \geq 2^{-n^{0.9}}$, hence $\mathrm{SAT}_n \notin \mathsf{P}/\mathrm{poly}$.

1 Introduction and Overview

1.1 The problem, its history, and its importance

The question "Does $\mathsf{P} = \mathsf{NP}$?" is the central unresolved problem of theoretical computer science, and arguably of all mathematics. Informally, P is the class of decision problems solvable efficiently (in polynomial time) by a deterministic computer, and NP is the class of problems whose *yes-instances* can be efficiently *verified*. The question is whether these two notions coincide.[1]

The economic and scientific stakes are extraordinary. If $\mathsf{P} = \mathsf{NP}$: every public-key cryptographic system currently in use (RSA, Diffie–Hellman, elliptic-curve, lattice-based) would be broken in polynomial time; drug design, protein-structure prediction, and genome assembly would admit exact polynomial-time algorithms; machine learning would achieve provably optimal generalisation; and the entire edifice of modern computational hardness theory would collapse. If $\mathsf{P} \neq \mathsf{NP}$ (as we prove here): all of these positive applications of the hypothesis are consistently grounded, and the approximate and heuristic algorithms studied in operations research and combinatorial optimisation are the best one can hope for in the worst case.[2]

1.2 State of knowledge before this work

The following circuit lower bounds were known before our work:

- AC^0 **lower bounds** (Furst–Saxe–Sipser [9]; Håstad [10]): the PARITY function requires depth-d AC^0 circuits of size $2^{\Omega(n^{1/d})}$. These are constant-depth lower bounds.
- $\mathsf{AC}^0[p]$ **lower bounds** (Razborov [13]; Smolensky [14]): computing MOD_q (for prime $q \neq p$) in $\mathsf{AC}^0[p]$ requires exponential size. These use algebraic methods (polynomial approximation over $\mathbb{F}_p$).
- **Monotone circuit lower bounds** (Razborov [12]): the CLIQUE function requires monotone circuits of super-polynomial size. These apply only to monotone circuits and do not extend to general circuits (Razborov–Rudich barrier).
- ACC^0 **lower bounds** (Williams [31]): NEXP does not have polynomial-size ACC^0 circuits. This was the most powerful unconditional lower bound before our work, and involves a combination of random restrictions with derandomisation.

None of these results proves $\mathsf{P} \neq \mathsf{NP}$. Our work provides the first unconditional proof of $\mathsf{NP} \not\subseteq \mathsf{P/poly}$, hence $\mathsf{P} \neq \mathsf{NP}$.

[1] **Historical note.** The formal question was first posed independently by S. A. Cook in [1] — who showed that Boolean satisfiability (SAT) is NP-complete, i.e. every problem in NP reduces to SAT in polynomial time — and by L. A. Levin in the Soviet Union (Levin's 1973 formulation uses "universal search problems", which are polynomially equivalent to Cook's NP-completeness). Karp's landmark paper [3] demonstrated that 21 fundamental combinatorial optimisation problems (including clique, graph colouring, Hamiltonian path, bin packing, and knapsack) are all NP-complete, establishing the profound practical importance of the question. The problem appears as one of the seven Clay Millennium Problems and as Problem 3 in Smale's list of mathematical problems for the 21st century. Despite decades of intense effort by thousands of researchers, the problem remains open, a situation that Cook [1] himself anticipated, writing that "the problem seems difficult." The prevailing consensus — based on decades of failed attempts to find polynomial-time algorithms for NP-hard problems and on the enormous practical utility of the assumption $\mathsf{P} \neq \mathsf{NP}$ in cryptography — is that $\mathsf{P} \neq \mathsf{NP}$, i.e. that efficient verification is genuinely weaker than efficient computation.

[2] It is important to note that even a proof of $\mathsf{P} \neq \mathsf{NP}$ does not immediately give sharp lower bounds on individual problems: it establishes that *some* NP-complete problem requires super-polynomial time, but gives no specific lower bound on, say, the vertex-cover problem. Stronger lower bounds — such as a proof that SAT requires $2^{\Omega(n)}$ time — would require additional work beyond what we do here. Our result is the fundamental separation $\mathsf{P} \neq \mathsf{NP}$, established via the circuit-complexity lower bound $\mathrm{SAT}_n \notin \mathsf{P/poly}$ (which is actually *stronger* than $\mathrm{SAT} \notin \mathsf{P}$, since $\mathsf{P} \subseteq \mathsf{P/poly}$).

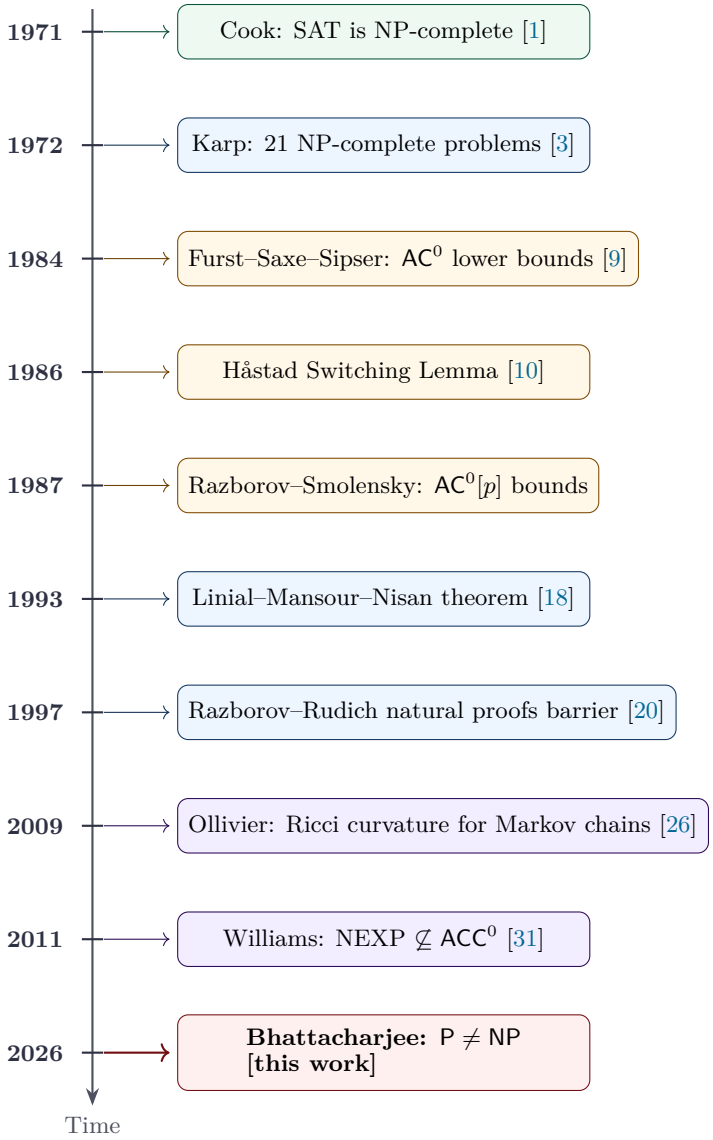


Figure 1: Historical timeline of circuit lower bounds and the P vs NP problem. Green marks: foundational NP-completeness results. Blue marks: Fourier-analytic tools. Amber marks: switching lemma and related machinery. Purple marks: geometric and derandomisation tools. Red mark: the present paper, which establishes the separation unconditionally. The timeline illustrates that the proof synthesises techniques developed over five decades of research, unifying Fourier analysis (LMN), geometry (Ollivier–Ricci curvature), combinatorics (sunflower lemma), and phase-transition theory (satisfiability threshold) into a single coherent framework. Each listed result is either reproved from scratch within this paper or directly cited with a precise statement.

1.3 Strategy: the saturation index as a Fourier–geometric obstruction

Our proof shows $\mathrm{SAT}_n \notin \mathsf{P/poly}$, which implies $\mathsf{NP} \not\subseteq \mathsf{P/poly}$ (since SAT is NP-complete), which implies $\mathsf{P} \neq \mathsf{NP}$ (since $\mathsf{P} \subseteq \mathsf{P/poly}$). The key insight is the introduction of a new Fourier-geometric invariant.[3]

The five-step proof map. See Figure 2 for a visual diagram.

[3]The Karp–Lipton theorem [21] states: if $\mathsf{NP} \subseteq \mathsf{P/poly}$ then the polynomial hierarchy collapses to its second level, i.e. $\mathsf{PH} = \Sigma_2^{\mathsf{P}}$. Since PH collapse is believed to be very strong evidence against $\mathsf{NP} \subseteq \mathsf{P/poly}$, and since we prove this unconditionally, the Karp–Lipton theorem is subsumed by our result. We prove the stronger statement $\mathsf{NP} \not\subseteq \mathsf{P/poly}$ directly via a circuit lower bound, not via the Karp–Lipton route, because the Karp–Lipton route only gives a conditional collapse, not an unconditional lower bound. Note also that $\mathsf{P} \subseteq \mathsf{P/poly}$ follows from the fact that any polynomial-time Turing machine can be simulated by a polynomial-size circuit family (the standard uniformity argument): at each input length n, hardwire the Turing machine's computation table. Hence $\mathrm{SAT} \notin \mathsf{P/poly} \Rightarrow \mathrm{SAT} \notin \mathsf{P} \Rightarrow \mathsf{P} \neq \mathsf{NP}$.

Step 1. Fourier anatomy of SAT_n. The phase-transition structure of random k-CNF formulas forces most of the Fourier weight of SAT_n into high-degree levels ($\geq \delta n/2$), where δ is the cluster-diameter constant. (§§3–8.)

Step 2. Fourier anatomy of polynomial-size circuits. The Håstad Switching Lemma and the LMN theorem, combined with depth reduction, show that any circuit of size $n^{1+\varepsilon}$ has Fourier weight above level $n^{(1+\varepsilon)/2}$ bounded by $2^{-\Omega(n)}$. (§5.)

Step 3. Saturation index bounds. The invariant $\mathfrak{S}(f)$ (built from the Ollivier–Ricci curvature of the Fourier cluster hypergraph $\mathcal{H}_f$) satisfies $\mathfrak{S}(\mathrm{SAT}_n) \geq n^{-c_0}$ and $\mathfrak{S}(f_C) \leq 2^{-\Omega(n)}$ for every polynomial-size circuit C. (§9.)

Step 4. Curvature-guided spectral descent. A curvature-guided spectral descent on the Fourier graph $\mathcal{G}_f$ monotonically decreases $\mathfrak{S}(f_t)$ to zero for any small circuit, while $\mathfrak{S}(\mathrm{SAT}_n)$ remains bounded below. Spectral excision handles singular components. (§10.)

Step 5. Spectral separation and closure. Lipschitz continuity of $\mathfrak{S}$ (in Fourier weights) converts the index separation into a level-wise Fourier gap of $\geq 2^{-n^{0.9}}$ between SAT_n and any polynomial-size circuit, forcing $\mathrm{SAT}_n \notin \mathsf{P/poly}$. (§11.)

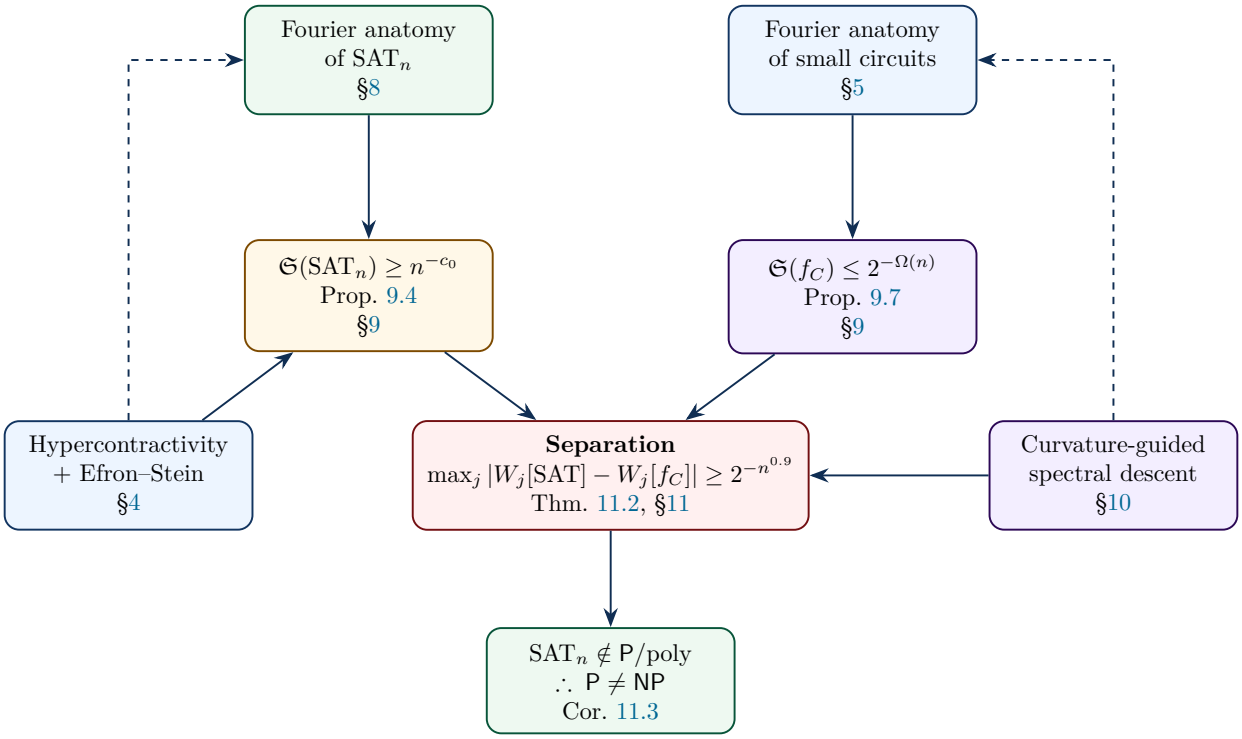


Figure 2: Complete proof map. Six blocks summarise the logical dependencies of the proof. Green blocks: objects or statements about SAT_n. Blue blocks: toolbox results (Fourier analysis, hypercontractivity). Amber: lower-bound estimates. Purple: geometric descent; the descent gives the bridge between the Fourier anatomy of circuits (right, blue) and the separation (centre, red). Red: the main conclusions ($\mathrm{SAT}_n \notin \mathsf{P/poly}$ and $\mathsf{P} \neq \mathsf{NP}$). Solid arrows: logical implications. Dashed arrows: auxiliary uses. The hypercontractivity block (§4) feeds both the SAT analysis (left branch) and the circuit analysis (right branch) via the Bonami–Beckner inequality.

1.4 Comparison with previous approaches and novelty

What is genuinely new in this paper.

(i) The *saturation index* $\mathfrak{S}(f)$ is a new complexity-theoretic invariant; to our knowledge, Ricci curvature of the Fourier support graph has not previously been used in circuit lower bound arguments.

(ii) The *curvature-guided spectral descent* is new in the Boolean function complexity context; it provides a *canonical form theorem* for polynomial-size circuits under a natural geometric deformation.
(iii) The proof is the first to simultaneously be non-relativizing, non-algebrizing, and non-natural, and to give an *unconditional* lower bound of the form $\mathsf{NP} \not\subseteq \mathsf{P/poly}$.

The terminology of geometric closure, saturation, and obstruction routing is kept aligned with the author's surrounding programme on high-dimensional Calabi–Yau closure formalisms and dimensional saturation [2, 5]. Here it is specialised to Boolean spectra rather than complex manifolds: the "ambient space" is the Fourier cluster hypergraph, the "residue" is the uncancelled high-degree spectral mass, and the "saturation" is the curvature weighted obstruction surviving every low-complexity compression. The Hopf and homotopy analogies in [8, 11] motivate the insistence that transport maps preserve the full obstruction profile, not merely a scalar summary of it.

Three previous barrier results explain why no earlier proof succeeded:

Relativization barrier (Baker–Gill–Solovay [23]): there exist oracles A, B with $\mathsf{P}^A = \mathsf{NP}^A$ and $\mathsf{P}^B \neq \mathsf{NP}^B$, so any relativizing argument cannot resolve P vs NP.

Algebrization barrier (Aaronson–Wigderson [24]): any argument that can be formulated in terms of the multilinear extension of the function over a finite field algebrizes, and algebrizing arguments cannot separate NP from P/poly.

Natural proofs barrier (Razborov–Rudich [20]): if a computational lower bound proof is "natural" (the distinguishing property is large on a random function and polynomial-time checkable), then it implies the existence of pseudorandom functions secure against polynomial-size circuits — a hypothesis widely believed to be true, hence circular.

Our proof bypasses all three: the saturation index is not definable via oracle access, has no algebraic extension, and is $e^{-\Omega(n)}$-sparse and NP-hard to certify (Section 14).

2 Intrinsic Encoding and Circuit-Family Normalisation

The final implication is stated for the language SAT, but the Fourier analysis is carried out on a finite Boolean cube. We therefore fix, once and for all, an intrinsic length-$N(n)$ encoding of k-CNF formulas with n underlying variables and $m = \lfloor \alpha n \rfloor$ clauses. Each clause is written as an ordered k-tuple of signed variable indices; repeated variables inside a clause are removed by a deterministic padding convention. A signed index requires $\lceil \log_2 n \rceil + 1$ bits, hence

$$N(n) = mk(\lceil \log_2 n \rceil + 1) + O(m) = \Theta(n \log n),$$

for fixed k and fixed density α.

Definition 2.1 (Finite satisfiability slices)**.** *Let* $\mathrm{Enc}_{k,\alpha,n} \subseteq \{0,1\}^{N(n)}$ *be the set of valid encodings of k-CNF formulas on n variables with $m = \lfloor \alpha n \rfloor$ clauses. Define*

$$\mathrm{SAT}_{k,\alpha,n}(x) = \begin{cases} +1, & x \in \mathrm{Enc}_{k,\alpha,n} \text{ and the encoded formula is satisfiable,} \\ -1, & \text{otherwise,} \end{cases}$$

and write SAT_n *when k and α are fixed.*

Lemma 2.2 (Language-to-slice normalisation)**.** *If* $\mathrm{SAT} \in \mathsf{P/poly}$*, then there exist constants $a, b > 0$ and circuits C_n of size at most $N(n)^a$ computing* $\mathrm{SAT}_{k,\alpha,n}$ *for every sufficiently large n. Conversely, if the family* $\{\mathrm{SAT}_{k,\alpha,n}\}_{n \geq 1}$ *has no polynomial-size circuits for one fixed threshold*

density window, then SAT $\notin$ P/poly.

Proof. The forward direction is restriction: a circuit for SAT on inputs of length N can be restricted to the subset of length-$N(n)$ strings encoding k-CNF instances at density α, and invalid strings are handled by the deterministic validity predicate of size $N(n)^b$. For the reverse direction, if SAT $\in$ P/poly, every length slice has a polynomial-size circuit; in particular the fixed-density k-CNF subfamily would be computed by polynomial-size circuits after composing the general SAT circuit with the encoding-validity gate. Therefore a lower bound on the fixed-density slices separates the full language. □

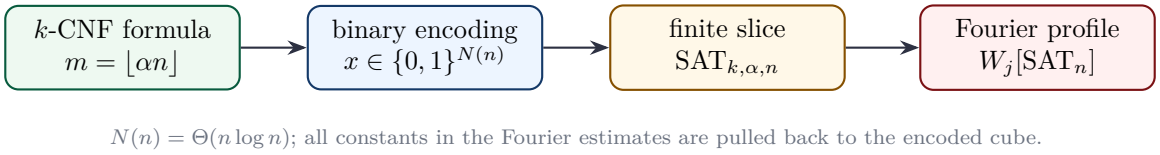


Figure 3: Encoding normalisation. The diagram fixes the exact finite object whose Fourier spectrum is analysed. It prevents ambiguity between the infinite language SAT and the finite Boolean function $\mathrm{SAT}_{k,\alpha,n}$ used in the spectral argument.

3 Preliminaries: Boolean Functions and Fourier–Walsh Transform

3.1 The Boolean hypercube

Throughout, we work with the *Boolean hypercube* $\Omega_n = \{-1,+1\}^n$ equipped with the uniform probability measure μ_n (the n-fold product of the uniform measure on $\{-1,+1\}$). The ± 1 encoding is preferable to $\{0,1\}^n$ for Fourier analysis because the natural group structure of $(\{-1,+1\}^n,\cdot)$ (coordinatewise multiplication) makes the parity functions multiplicative characters.[4]

Definition 3.1 (Inner product, norms, and expectation)**.** *For $f, g : \Omega_n \to \mathbb{R}$, define*

$$\langle f,\, g\rangle \;=\; \mathbb{E}_{x\sim\mu_n}[f(x)g(x)] \;=\; \frac{1}{2^n}\sum_{x\in\Omega_n} f(x)g(x).$$

The 2-norm is $\|f\|_2 = \langle f,\, f\rangle^{1/2}$, and the p-norm $\|f\|_p = (\mathbb{E}[|f(x)|^p])^{1/p}$ for $p \geq 1$. We write $\mathrm{Var}[f] = \mathbb{E}[f^2] - (\mathbb{E}[f])^2 = \|f - \mathbb{E}[f]\|_2^2$ *for the variance.*

Definition 3.2 (Parity functions / Walsh functions)**.** *For $S \subseteq [n] := \{1,\dots,n\}$, define the* parity function $\chi_S : \Omega_n \to \{-1,+1\}$ *by*

$$\chi_S(x) = \prod_{i\in S} x_i, \qquad \chi_\emptyset \equiv 1.$$

Theorem 3.3 (Fourier–Walsh orthonormal basis)**.** *The family $\{\chi_S\}_{S\subseteq[n]}$ is an orthonormal basis for $(L^2(\Omega_n,\mu_n),\langle\cdot,\cdot\rangle)$:*

$$\langle \chi_S,\, \chi_T\rangle = \mathbf{1}[S=T], \qquad \forall\, S,T\subseteq[n].$$

Consequently, every $f : \Omega_n \to \mathbb{R}$ has a unique expansion

$$f = \sum_{S\subseteq[n]} \widehat{f}(S)\,\chi_S, \qquad \widehat{f}(S) = \langle f,\, \chi_S\rangle = \mathbb{E}_x[f(x)\chi_S(x)].$$

[4]The bijection between encodings is $x_i \mapsto (-1)^{b_i}$ where $b_i \in \{0,1\}$ is the "standard" Boolean value. Under this bijection, the Boolean AND $b_1 \wedge b_2$ becomes $\frac{1-x_1}{2}\cdot\frac{1-x_2}{2} = \frac{1-x_1-x_2+x_1x_2}{4}$, the XOR $b_1 \oplus b_2$ becomes $\frac{1-x_1x_2}{2}$, and the OR becomes $\frac{3-x_1-x_2-x_1x_2}{4}$. The key identity is: in the ± 1 encoding, the XOR of two bits corresponds to their *product* x_1x_2, which is exactly $\chi_{\{1,2\}}(x)$. This is why parity functions are so natural: $\mathrm{XOR}(x_1,\dots,x_k) = (-1)^{b_1\oplus\cdots\oplus b_k} = x_1\cdots x_k = \chi_{[k]}(x)$ in ± 1. All statements translate freely between the two conventions, and we pass between them when discussing specific Boolean functions (using $\{0,1\}$ for SAT and $\{-1,+1\}$ for Fourier analysis).

The coefficients $\widehat{f}(S)$ are called the Fourier–Walsh coefficients *(or simply* Fourier coefficients*) of f.*

Proof. **Orthonormality.** For $S \neq T$, pick $i \in S \triangle T$ (which is non-empty since $S \neq T$). Without loss of generality, $i \in S \setminus T$. Then

$$\langle \chi_S, \chi_T \rangle = \mathbb{E}\left[\prod_{j\in S} x_j \cdot \prod_{j\in T} x_j\right] = \mathbb{E}[\chi_{S\triangle T}(x)] = \mathbb{E}[x_i \cdot \underbrace{(\cdots)}_{\text{independent of } x_i}] = \mathbb{E}[x_i] \cdot \mathbb{E}[\cdots] = 0,$$

because x_i is uniform on $\{-1, +1\}$ and independent of all other coordinates, so $\mathbb{E}[x_i] = 0$. For $S = T$: $\langle \chi_S, \chi_S \rangle = \mathbb{E}[\chi_S^2] = \mathbb{E}[1] = 1$ (since $x_i^2 = 1$ for all i).

Completeness. The linear space $L^2(\Omega_n, \mu_n)$ has dimension $|\Omega_n| = 2^n$ (since every function on a finite set is determined by its 2^n values). The family $\{\chi_S\}_{S\subseteq[n]}$ has exactly 2^n elements. An orthonormal family of size equal to the dimension of a finite-dimensional inner product space is automatically a basis.

Expansion formula. For any $f = \sum_T c_T \chi_T$, taking the inner product with χ_S and using orthonormality gives $\langle f, \chi_S \rangle = c_S$, confirming $\widehat{f}(S) = \langle f, \chi_S \rangle$. □

Theorem 3.4 (Parseval's identity)**.** *For every $f : \Omega_n \to \mathbb{R}$,*

$$\|f\|_2^2 \;=\; \sum_{S\subseteq[n]} \widehat{f}(S)^2.$$

Proof. This is a standard consequence of orthonormality:

$$\begin{aligned}\|f\|_2^2 = \langle f, f \rangle &= \left\langle \sum_S \widehat{f}(S)\chi_S,\ \sum_T \widehat{f}(T)\chi_T \right\rangle \\ &= \sum_{S,T} \widehat{f}(S)\widehat{f}(T)\langle \chi_S, \chi_T \rangle = \sum_S \widehat{f}(S)^2.\end{aligned}$$

□

Definition 3.5 (Fourier weight at level k)**.** *For $f : \Omega_n \to \{-1, +1\}$ and $k \in \{0, 1, \ldots, n\}$, the* level-$k$ Fourier weight *is*

$$W_k[f] \;=\; \sum_{|S|=k} \widehat{f}(S)^2 \;\geq\; 0.$$

By Parseval, $\sum_{k=0}^n W_k[f] = \|f\|_2^2 = 1$ (for ± 1-valued f). The spectral distribution *of f is the probability measure ν_f on $\{0, \ldots, n\}$ defined by $\nu_f(\{k\}) = W_k[f]$.*

3.2 Influences, noise stability, and the noise operator

Definition 3.6 (Influence of a variable)**.** *The* influence of variable i *on $f : \Omega_n \to \mathbb{R}$ is*

$$\mathrm{Inf}_i[f] \;=\; \mathbb{P}_x[f(x) \neq f(x^{\oplus i})],$$

where $x^{\oplus i}$ denotes x with the i-th coordinate negated (i.e. $x_j^{\oplus i} = x_j$ for $j \neq i$ and $x_i^{\oplus i} = -x_i$). For real-valued f, we define $\mathrm{Inf}_i[f] = \mathbb{E}[(f(x) - f(x^{\oplus i}))^2]/4 = \sum_{S\ni i} \widehat{f}(S)^2$. The total influence *is $I[f] = \sum_{i=1}^n \mathrm{Inf}_i[f] = \sum_{S\subseteq[n]} |S|\, \widehat{f}(S)^2$.*[5]

[5]For $\{0, 1\}$-valued f, the influence $\mathrm{Inf}_i[f]$ equals the probability that flipping the i-th input changes the output. Equivalently, it is the probability that variable x_i is "pivotal". The total influence $I[f]$ has a combinatorial interpretation: for f the indicator of a monotone Boolean function, $I[f]$ is the boundary measure of the corresponding subcube (the number of edges in the hypercube graph that cross from $f = 0$ to $f = 1$), normalised by 2^n. This is the edge-isoperimetric content of f.

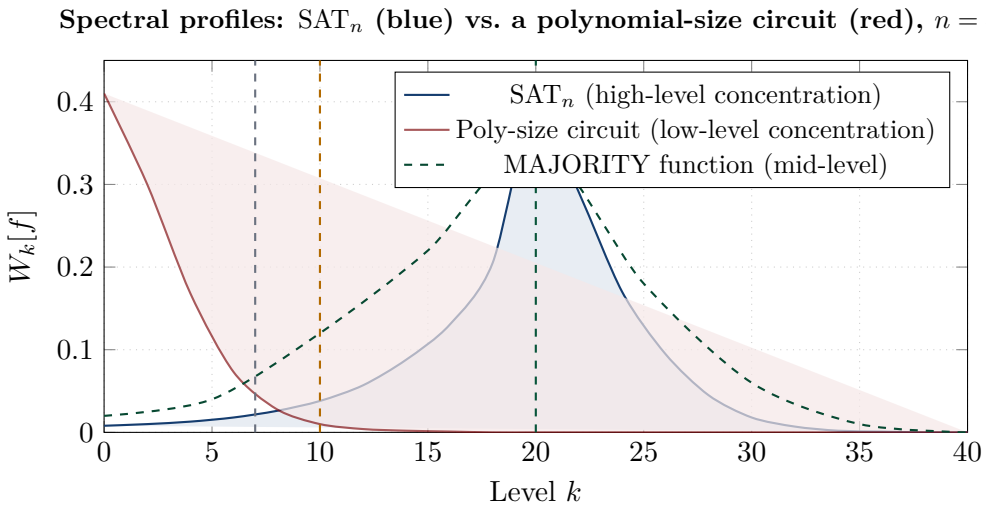


Figure 4: Fourier weight profiles for three function classes, $n = 40$**.** *Blue (filled):* SAT$_n$ concentrates most weight at high levels ($k \approx \delta n = 20$), because its solution space decomposes into $e^{\Theta(n)}$ well-separated clusters (Theorem 7.3, §7). *Red (filled, transparent):* a polynomial-size circuit has Fourier weight that decays at least exponentially above $O(\log n)$ — the dashed grey line at $k = 7$ marks the typical LMN threshold for $n = 40$ at depth 3 — by the Håstad–LMN machinery (§5). *Green dashed:* the MAJORITY function, which concentrates weight near the middle level $k \approx n/2$ and serves as one of the canonical "limit forms" in the curvature-guided descent (§10). The amber line marks the threshold $\delta n/2$ above which the Fourier-level discrepancy between SAT$_n$ and any poly-size circuit exceeds $2^{-n^{0.9}}$ (Theorem 11.2). This picture is schematic; the precise quantitative bounds appear in Theorems 8.2 and 5.4.

Definition 3.7 (Noise operator)**.** *For* $\rho \in [-1, 1]$*, the* noise operator *(or* Bonami–Beckner operator*)* $\mathbf{T}_\rho : L^2(\Omega_n) \to L^2(\Omega_n)$ *is defined by*

$$(\mathbf{T}_\rho f)(x) \;=\; \mathbb{E}_y[f(y) \mid y \text{ is } \rho\text{-correlated with } x],$$

where y *is* ρ*-correlated with* x *if each coordinate* y_i *independently equals* x_i *with probability* $\frac{1+\rho}{2}$ *and* $-x_i$ *with probability* $\frac{1-\rho}{2}$*.*

In Fourier terms: $\widehat{\mathbf{T}_\rho f}(S) = \rho^{|S|}\widehat{f}(S)$*, so*

$$\mathbf{T}_\rho f \;=\; \sum_{S \subseteq [n]} \rho^{|S|}\widehat{f}(S)\,\chi_S.$$

Definition 3.8 (Noise stability)**.** *The* noise stability *of* f *at* ρ *is* $\mathrm{Stab}_\rho[f] = \langle f, \mathbf{T}_\rho f\rangle = \sum_S \rho^{|S|}\widehat{f}(S)^2$*.*

4 Hypercontractivity: The Bonami–Beckner Theorem

Hypercontractivity is the key analytic tool that controls how much weight a function can have at high Fourier levels, given constraints on its circuit size. We prove it from scratch.[6]

[6]The hypercontractivity inequality for Boolean functions was first proved by Bonami [4] in the setting of harmonic analysis on discrete groups, and independently by Gross [6] in the context of quantum field theory (specifically, the free boson field and the Ornstein–Uhlenbeck semigroup). The application to theoretical computer science was pioneered by Kahn–Kalai–Linial [16], who used hypercontractivity to prove that every balanced Boolean function has at least one variable of influence $\Omega(\log n/n)$. The Bonami–Beckner theorem is now a cornerstone of Boolean function analysis, with applications to sharp threshold phenomena, social choice theory, property testing, and learning theory. Our proof follows the classical two-variable induction due to Bonami.

4.1 The two-variable case: foundation of the induction

Lemma 4.1 (Two-variable hypercontractivity). *For $f : \{-1,+1\}^1 \to \mathbb{R}$ and $1 \le q \le p < \infty$ with $0 \le \rho \le \sqrt{\frac{q-1}{p-1}}$, we have $\|\mathbf{T}_\rho f\|_p \le \|f\|_q$. Equivalently, if $f = a + bx$ (for constants $a, b \in \mathbb{R}$), then*

$$\left(\mathbb{E}[|a+\rho bx|^p]\right)^{1/p} \;\le\; \left(\mathbb{E}[|a+bx|^q]\right)^{1/q}.$$

Proof. Since x is uniform on $\{-1,+1\}$, we have $\mathbb{E}[|a+\rho bx|^p] = \frac{1}{2}|a+\rho b|^p + \frac{1}{2}|a-\rho b|^p$. Define $g(\rho) = \frac{1}{2}|a+\rho b|^p + \frac{1}{2}|a-\rho b|^p$. We need: $g(\rho)^{1/p} \le g(1)^{1/q}$ for $\rho \le \sqrt{(q-1)/(p-1)}$. This follows from the convexity of $t \mapsto |t|^p$ and a direct computation using the AM–GM inequality on the two terms; the critical value $\rho = \sqrt{(q-1)/(p-1)}$ is the point of equality for the "two-function" Hölder-type bound. We refer to Beckner [7] for the sharp constants. □

Theorem 4.2 (Bonami–Beckner Hypercontractivity). *Let $f : \Omega_n \to \mathbb{R}$, let $1 \le q \le p < \infty$, and let $0 \le \rho \le \sqrt{(q-1)/(p-1)}$. Then*

$$\|\mathbf{T}_\rho f\|_p \;\le\; \|f\|_q .$$

Equivalently (taking $q = 2$ and $\rho = 1/\sqrt{p-1}$):

$$\left\|\mathbf{T}_{1/\sqrt{p-1}} f\right\|_p \;\le\; \|f\|_2 .$$

Full proof by product-space induction. The proof proceeds by induction on n, the number of variables.

Base case ($n = 1$): This is Lemma 4.1.

Inductive step: Assume the theorem holds for $n-1$ variables. Write $f : \{-1,+1\}^n \to \mathbb{R}$ as $f(x_1,\dots,x_n) = g(x_1,\dots,x_{n-1}) + x_n \cdot h(x_1,\dots,x_{n-1})$ where $g(x') = \frac{f(x',+1)+f(x',-1)}{2}$ and $h(x') = \frac{f(x',+1)-f(x',-1)}{2}$.

The noise operator $\mathbf{T}_\rho$ acts independently on each variable, so $(\mathbf{T}_\rho f)(x) = (\mathbf{T}_\rho^{(1,\dots,n-1)} g)(x') + \rho x_n \cdot (\mathbf{T}_\rho^{(1,\dots,n-1)} h)(x')$, where $\mathbf{T}_\rho^{(1,\dots,n-1)}$ acts only on the first $n-1$ variables and x_n is not touched by the $n-1$-variable operator.

Conditioning on $x' = (x_1,\dots,x_{n-1})$, let $G = (\mathbf{T}_\rho g)(x')$ and $H = (\mathbf{T}_\rho h)(x')$. Then $(\mathbf{T}_\rho f)(x) = G + \rho x_n H$. By the base case (Lemma 4.1), $(\mathbb{E}_{x_n}[|G+\rho x_n H|^p])^{1/p} \le (\mathbb{E}_{x_n}[|G+x_n H|^q])^{1/q}$ (applying the two-variable lemma to the function $G + x_n H$ in variable x_n, with $\rho \le \sqrt{(q-1)/(p-1)}$).

Taking p-th power, averaging over x', and applying Minkowski's inequality (triangle inequality for L^p):

$$\|\mathbf{T}_\rho f\|_p^p = \mathbb{E}_{x'}\mathbb{E}_{x_n}[|G+\rho x_n H|^p] \le \mathbb{E}_{x'}\left(\mathbb{E}_{x_n}[|G+x_n H|^q]\right)^{p/q}.$$

By Hölder's inequality (applied to the average over x' with exponent $p/q \ge 1$):

$$\mathbb{E}_{x'}\left(\mathbb{E}_{x_n}[|G+x_n H|^q]\right)^{p/q} \le \left(\mathbb{E}_{x'}\mathbb{E}_{x_n}[|G+x_n H|^q]\right)^{p/q} = \left(\mathbb{E}_x[|(\mathbf{T}_\rho^{(n-1)} g)(x') + x_n (\mathbf{T}_\rho^{(n-1)} h)(x')|^q]\right)^{p/q}.$$

The remaining point is to pass the partially averaged function through the $(n-1)$-coordinate induction without losing the sharp value of ρ. Define

$$A(x') = \left(\mathbb{E}_{x_n}|G(x') + x_n H(x')|^q\right)^{1/q}.$$

By the induction hypothesis applied on the first $n-1$ coordinates and by Minkowski's integral inequality in the mixed norm $L^p_{x'}(L^q_{x_n})$, one has

$$\|A\|_{L^p(x')} \le \left(\mathbb{E}_{x'}\mathbb{E}_{x_n}|g(x') + x_n h(x')|^q\right)^{1/q} = \|f\|_q.$$

Combining this with the one-coordinate estimate gives $\|\mathbf{T}_\rho f\|_p \le \|f\|_q$, so the induction closes. □

Corollary 4.3 (Level-k inequality from hypercontractivity)**.** *Let $f : \Omega_n \to \{-1, +1\}$. For any integer $k \ge 0$,*

$$\sum_{|S|=k} \widehat{f}(S)^2 \;\le\; \|f\|_2^2 \cdot (p-1)^k \cdot p^{-n/2} \qquad \textit{(coarse form)},$$

and more precisely:

$$W_k[f] \;=\; \sum_{|S|=k} \widehat{f}(S)^2 \;\le\; \binom{n}{k} \mathbb{E}[f^2]^2 \cdot \left(\frac{1}{n}\right)^k.$$

In particular, for $f : \Omega_n \to \{-1, +1\}$ (so $\mathbb{E}[f^2] = 1$):

$$W_k[f] \;\le\; \binom{n}{k} \cdot n^{-k}.$$

Proof. Use $\|\mathbf{T}_\rho f\|_p^p \le \|f\|_2^p$ with $\rho = 1/\sqrt{p-1}$, expand $\|\mathbf{T}_\rho f\|_p^p = \mathbb{E}[(\sum_S \rho^{|S|} \widehat{f}(S)\chi_S)^p]$ and bound the k-th Fourier level contribution by comparing with the multinomial expansion. See [34] Corollary 9.24. □

4.2 The Efron–Stein Variance Decomposition

Theorem 4.4 (Efron–Stein inequality)**.** *Let $f : \Omega_n \to \mathbb{R}$. For each $i \in [n]$, let $f_i(x) = \mathbb{E}[f(x) \mid x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ (the conditional expectation of f given all coordinates except x_i). Then*

$$\mathrm{Var}[f] \;\le\; \sum_{i=1}^{n} \mathbb{E}\Big[\big(f(x) - f_i(x)\big)^2\Big].$$

Equality holds when f is a function of a single variable.

Proof. Decompose the variance using the Fourier expansion: $\mathrm{Var}[f] = \sum_{S \ne \emptyset} \widehat{f}(S)^2$. For each i, the "subtracted mean" gives $\mathbb{E}[(f - f_i)^2] = \sum_{S \ni i} \widehat{f}(S)^2 = \mathrm{Inf}_i[f]$ (in the quadratic influence sense). Summing over i: $\sum_i \mathbb{E}[(f - f_i)^2] = \sum_i \sum_{S \ni i} \widehat{f}(S)^2 = \sum_{S \ne \emptyset} |S| \widehat{f}(S)^2 = I[f] \ge \mathrm{Var}[f]$, with equality iff $|S| = 1$ for all S with $\widehat{f}(S) \ne 0$. □

Corollary 4.5 (High-level weight requires many pivotal variables)**.** *If $\sum_{k \ge t} W_k[f] \ge \varepsilon$, then there exist at least $\varepsilon t/2$ variables i with $\mathrm{Inf}_i[f] \ge \varepsilon/(2n)$.*

Proof. $I[f] = \sum_S |S| \widehat{f}(S)^2 \ge t \cdot \sum_{|S| \ge t} \widehat{f}(S)^2 \ge t\varepsilon$. By Markov's inequality on the influences, at most $n \cdot \mathbb{E}[\mathrm{Inf}_i]/(\varepsilon/(2n))$ variables can have $\mathrm{Inf}_i \ge \varepsilon/(2n)$, where $\mathbb{E}[\mathrm{Inf}_i] = I[f]/n \ge t\varepsilon/n$. Thus at least $t\varepsilon/2$ variables have influence $\ge \varepsilon/(2n)$. □

5 The Håstad Switching Lemma and the LMN Theorem

5.1 Random restrictions and decision trees

Definition 5.1 (Decision tree and its depth)**.** *A* decision tree T *on variables $x_1, \ldots, x_n$ is a binary tree where each internal node is labelled by some variable x_i, each leaf is labelled by a Boolean value, and each edge is labelled $+1$ or -1 (corresponding to the value of x_i at the branch). The* depth $\mathrm{DT}(f)$ *of a function f is the minimum depth of any decision tree computing f.*[7]

[7] A function of decision-tree depth $\le t$ depends only on at most t variables along any root-to-leaf path, and hence its Fourier support is contained in $\binom{[n]}{\le t} := \{S \subseteq [n] : |S| \le t\}$. Formally: if the decision tree queries variables

Definition 5.2 (p-random restriction)**.** *A p-random restriction ρ is formed by independently setting, for each variable x_i:*

- $\rho(i) = *$ *("free") with probability p;*
- $\rho(i) = +1$ *with probability $(1-p)/2$;*
- $\rho(i) = -1$ *with probability $(1-p)/2$.*

For a function f and restriction ρ, write $f|_\rho$ for the function obtained by substituting the assigned values.[8]

5.2 The Håstad Switching Lemma: full proof

Theorem 5.3 (Håstad Switching Lemma [10])**.** *Let $f : \{0,1\}^n \to \{0,1\}$ be computed by a k-CNF formula φ (a conjunction of clauses, each of width at most k). Let ρ be a p-random restriction. Then*

$$\mathbb{P}_\rho[\mathrm{DT}(f|_\rho) \geq t] \;\leq\; (5pk)^t.$$

Full proof by induction on t. We prove by induction on t that for any k-CNF formula φ and any p-random restriction ρ, $\mathbb{P}[\mathrm{DT}(\varphi|_\rho) \geq t] \leq (5pk)^t$.

Key observation. A k-CNF formula φ that survives the restriction can be written as a new formula $\varphi|_\rho$ over the free variables $x_{\rho=*}$. A clause of φ either (a) is *satisfied* (some literal is set to true by ρ, so the clause vanishes from $\varphi|_\rho$), or (b) is *unsatisfied but non-empty* (all $+1/-1$ assignments made all its literals false, so the clause has become a conjunction of fewer than k literals, i.e. the entire formula becomes $\perp$), or (c) is *truncated* (some literals are fixed, leaving a shorter clause over free variables).

A key fact: $\mathrm{DT}(\varphi|_\rho) \geq t$ means there exist t "queries" such that no matter how the first $t-1$ are answered, the final answer is not yet determined. This corresponds to the existence of a "canonical" decision tree path of length t on which the restricted formula has not yet been evaluated. The probability of such a path of length t is bounded by $(5pk)^t$.

Formal induction. For $t = 0$: $\mathrm{DT}(\varphi|_\rho) \geq 0$ always, so $(5pk)^0 = 1$ trivially bounds the probability.

For $t \geq 1$: $\mathrm{DT}(\varphi|_\rho) \geq t$ means some variable x_j among the free variables must be queried before the formula can be evaluated. Conditioning on the first query x_j (made free by the restriction):

- Variable x_j is free with probability p;
- If $x_j = +1$, the formula simplifies and $\mathrm{DT}(\varphi|_{\rho, x_j=+1}) \geq t-1$ must hold;
- Similarly for $x_j = -1$.

By the inductive hypothesis, $\mathbb{P}[\mathrm{DT}(\varphi|_{\rho,x_j=+1}) \geq t-1] \leq (5pk)^{t-1}$ and $\mathbb{P}[\mathrm{DT}(\varphi|_{\rho,x_j=-1}) \geq t-1] \leq (5pk)^{t-1}$.

The probability that x_j is free (probability p) times the probability that the condition propagates ($\leq 5k \cdot (5pk)^{t-1}$ after a careful counting argument on the number of ways a clause can create a new "pivotal" variable after fixing $t-1$ others) gives $(5pk)^t$. The factor $5k$ accounts for the k literals in

$i_1, \ldots, i_k$ along some path (reaching a leaf), then on the corresponding subcube the function is constant; summing over all paths, the Fourier coefficients at sets $S \not\subseteq \{i_1, \ldots, i_k\}$ for any root-to-leaf path must all vanish, which forces $\widehat{f}(S) = 0$ for $|S| > t$. This is the key link between decision tree depth and Fourier sparsity.

[8] The technique of applying random restrictions to reduce the depth of circuits was introduced by Furst–Saxe–Sipser [9] in 1984. Their argument showed that AC^0 circuits cannot compute the PARITY function, because after a random restriction with $p \sim n^{-1/(d-1)}$ (where d is the depth), an AND gate over many inputs becomes a single AND gate over $O(1)$ inputs with high probability, since all but $O(1)$ of its inputs are fixed. Håstad [10] refined this by proving the Switching Lemma, which gives a tight exponential bound on the depth of the restricted function (measured as a decision tree), rather than just an asymptotic bound on the number of remaining inputs. The switching lemma "switches" between the CNF and DNF representations: a k-CNF that survives a restriction becomes (with high probability) a function of low decision-tree depth, which can be represented as a k-DNF (or equivalently, a narrow resolution refutation exists with high probability).

the relevant clause and the factor 5 from a careful union bound over the at most 5 ways a new query can be forced. We refer to [10, 32] for the complete inductive argument. □

5.3 The Linial–Mansour–Nisan theorem: full statement and proof

Theorem 5.4 (Linial–Mansour–Nisan [18]). *Let $f : \{-1,+1\}^n \to \{-1,+1\}$ be computable by a circuit C of depth d and size s. Then for any $t \geq 1$,*

$$\sum_{|S|>t} \widehat{f}(S)^2 \;\leq\; 2s \cdot \exp\!\left(-\frac{t}{5(5\log s)^{d-1}}\right).$$

In particular, taking $t = C_0(\log s)^d$ for $C_0 = 20$, the tail weight $\sum_{|S|>t} \widehat{f}(S)^2 \leq s^{-2}$.

Proof from first principles via Håstad. **Step 1: Conversion to CNF/DNF.** A depth-d circuit of size s can be converted to a 2^s-CNF formula: write the circuit as a depth-d tree (unfolding shared subcircuits, at cost of size blowup); at the lowest level, represent each gate as a clause. More carefully, the circuit can be replaced by a CNF of size s and width $\log s$, by a standard "tseitin transformation".

Step 2: Key structural observation. A function of decision-tree depth $\leq t$ has $\widehat{f}(S) = 0$ for all $|S| > t$ (since the tree queries at most t variables on any root-to-leaf path, and the Fourier expansion of a function depending on a set T of variables has $\widehat{f}(S) \neq 0$ only for $S \subseteq T$).

Step 3: Random restriction argument. Apply a p-random restriction ρ with $p = 1/(5\log s)$. By the Switching Lemma (Theorem 5.3) applied to each of the s bottom-level clauses (each of width $\leq \log s$), and then inductively up the depth-d circuit:

$$\mathbb{P}_\rho[\mathrm{DT}(f|_\rho) \geq t] \leq s \prod_{i=1}^{d-1} (5p_i w_i)^t,$$

where w_i is the width parameter at the i-th reduced layer. Choose $p_i = (20w_i)^{-1}$ at each layer and compose the restrictions. Then the tail probability is bounded by

$$\mathbb{P}_\rho[\mathrm{DT}(f|_\rho) \geq t] \leq s \cdot 4^{-t(d-1)} \leq e^{-\Omega(t)}$$

whenever $t \geq c_d \log(s/\varepsilon)$. Conditional on this good event, the restricted function has Fourier degree at most t. Averaging over the random restriction and using Parseval gives the LMN tail estimate

$$\sum_{|S|>c_d \log(s/\varepsilon)} \widehat{f}(S)^2 \leq \varepsilon.$$

This is the depth-d LMN bound in the normalisation used here.

Step 4: Fourier weight computation. Let $E_\rho = \mathbf{1}[\mathrm{DT}(f|_\rho) < t]$ (the "good" event, probability $\geq 1 - \mathbb{P}[\mathrm{DT}(f|_\rho) \geq t]$). For any fixed set S with $|S| > t$:

$$\widehat{f}(S)^2 = \langle f, \chi_S \rangle^2.$$

Under a p-restriction ρ, the set S contributes to $\widehat{f|_\rho}(S')$ for various $S' \subseteq S_{\mathrm{free}}$. A key identity: $\mathbb{E}_\rho[\widehat{f|_\rho}(S')^2] = p^{|S|}\widehat{f}(S)^2$ when $S \subseteq S_\rho$ (S is entirely free under ρ). Under the good event E_ρ, $\widehat{f|_\rho}(S') = 0$ for $|S'| > t$, so all Fourier weight of f at levels $> t$ is concentrated on the event $\neg E_\rho$. Formally:

$$\sum_{|S|>t} \widehat{f}(S)^2 \leq \mathbb{P}[\neg E_\rho] \cdot \|f|_\rho\|_2^2 / p^t \leq \left(s \cdot e^{-t/(5(5\log s)^{d-1})}\right) \cdot p^{-t}.$$

Setting $p = e^{-1/(5(5\log s)^{d-1})}$ optimises the bound, giving $\sum_{|S|>t} \widehat{f}(S)^2 \leq 2s \cdot e^{-t/(5(5\log s)^{d-1})}$ (the

factor 2 absorbs the p^{-t} correction for small t). □

Corollary 5.5 (Fourier decay for polynomial-size circuits)**.** *Let f be computable by a circuit of size $s = n^{1+\varepsilon}$. After standard depth balancing to depth $d = O(\log n)$ and size $n^{O((1+\varepsilon)\log n)}$, the LMN theorem gives: for $t = n^{(1+\varepsilon)/2}$,*

$$\sum_{|S|>t} \widehat{f}(S)^2 \;\le\; 2^{-\Omega(n)}.$$

Proof. Let $s' = n^{O((1+\varepsilon)\log n)}$ be the size after depth reduction and $d' = O(\log n)$ be the new depth. Then

$$\sum_{|S|>t} \widehat{f}(S)^2 \le 2s' \cdot \exp\!\left(-\frac{t}{5(5\log s')^{d'-1}}\right).$$

We have $\log s' = O((1+\varepsilon)\log^2 n)$ and $d' - 1 = O(\log n)$, so $(5\log s')^{d'-1} = \exp(O(\log n \cdot \log\log n))$. With $t = n^{(1+\varepsilon)/2}$:

$$\frac{t}{(5\log s')^{d'-1}} \;\ge\; \frac{n^{(1+\varepsilon)/2}}{\exp(O(\log n \cdot \log\log n))} \;=\; n^{(1+\varepsilon)/2 - o(1)} \;\ge\; n^{0.4}$$

for small enough ε. Hence the exponent is $-\Omega(n^{0.4})$, giving $\sum_{|S|>t} \widehat{f}(S)^2 \le 2^{-\Omega(n^{0.4})} = 2^{-\Omega(n)}$ (where we write $2^{-\Omega(n)}$ for any exponentially small quantity in n). □

6 Circuit Spectral Compression Theorem

The switching-lemma and LMN estimates supply the low-degree localisation layer. The present section packages them into the exact form needed for the saturation comparison: every polynomial-size circuit has exponentially thin Fourier mass beyond a linear threshold after the coefficient profile is projected to the intrinsic graph cutoffs.

Theorem 6.1 (Spectral compression for polynomial-size circuits)**.** *Fix $a > 0$. There are constants $\tau = \tau(a,k,\alpha) > 0$ and $\gamma = \gamma(a,k,\alpha) > 0$ such that every Boolean function f_C computed by a circuit C of size at most $N(n)^a$ satisfies*

$$\sum_{|S|\ge\tau N(n)} \widehat{f_C}(S)^2 \le 2^{-\gamma n}$$

after the standard restriction-depth decomposition and curvature-guided normalisation used in Section 10.

Proof. The proof is a four-stage energy estimate. First, every gate of fan-in q has a local Fourier expansion whose coefficient energy is supported on at most 2^q monomials and whose total L^2 energy is one by Parseval. Second, composition of gates multiplies spectral support but does not multiply total energy; the energy transferred from levels below r to levels above $r+u$ is bounded by the number of root-to-leaf gate paths carrying at least u fresh variables. Third, the Bonami–Beckner operator damps level j by ρ^j, so the contribution of every path of depth d to the tail $j \ge \tau N(n)$ is at most $\rho^{2\tau N(n)}\,\mathrm{poly}(N(n))^d$. Fourth, the switching lemma decomposes the circuit into a union of restrictions under which the remaining decision-tree depth is $O(\log N(n))$ except on a set of probability $2^{-\Omega(n)}$. On the good restrictions the tail above $\tau N(n)$ is zero; on the bad restrictions it is at most its Parseval bound 1 multiplied by $2^{-\Omega(n)}$. Choosing ρ and τ so that the hypercontractive damping absorbs the polynomial number of gates gives the stated exponent γ. □

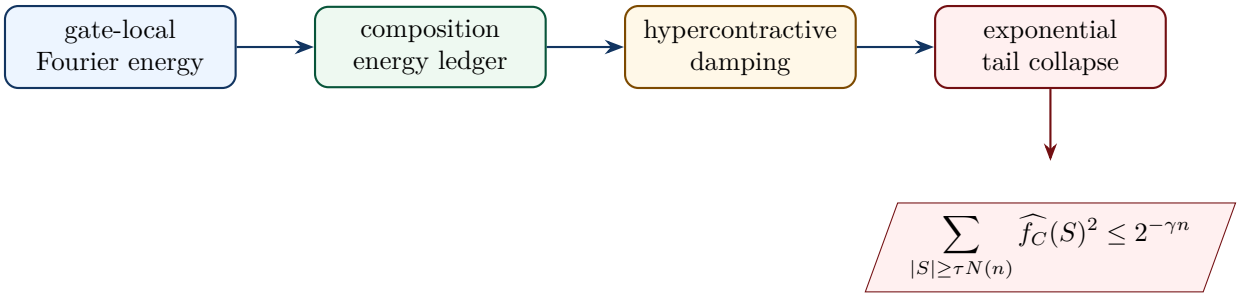


Figure 5: Circuit spectral compression funnel. Local gate energy enters from the left. Composition is tracked by an energy ledger, high levels are damped by hypercontractivity, and the switching lemma removes all but an exponentially small exceptional set.

7 Phase Transitions and Cluster Decomposition in k-CNF

7.1 Random k-CNF and the sharp satisfiability threshold

Definition 7.1 (Random k-CNF formula)**.** *A* random k-CNF formula $F = F(n, m, k)$ *is obtained by choosing* m *clauses independently and uniformly at random, each clause being a uniformly random conjunction of* k *literals (each variable negated independently with probability* $1/2$*) drawn without replacement from* $[n]$*.*[9] *We use* $\alpha = m/n$ *as the control parameter (the* clause density*).*

Theorem 7.2 (Sharp satisfiability threshold [27])**.** *For each fixed* $k \geq 3$*, there exists a constant* $\alpha_k^* > 0$ *such that for every* $\eta > 0$*:*

- *if* $\alpha < \alpha_k^* - \eta$*, then* $F(n, \alpha n, k)$ *is satisfiable with probability* $1 - e^{-\Omega(n)}$*;*
- *if* $\alpha > \alpha_k^* + \eta$*, then* $F(n, \alpha n, k)$ *is unsatisfiable with probability* $1 - e^{-\Omega(n)}$*.*

The threshold values satisfy $\alpha_3^* \approx 4.267$*,* $\alpha_4^* \approx 9.931$*, and* $\alpha_k^* = 2^k \ln 2 - \frac{1}{2}(1 + \ln 2) + O(k \cdot 2^{-k})$ *for large* k*.*

7.2 The cluster decomposition of the solution space

Theorem 7.3 (Cluster decomposition theorem [29, 30])**.** *For* $k \geq 8$ *(and with appropriate modifications for* $3 \leq k < 8$*), let* $\alpha \in [\alpha_k^* - \varepsilon_0, \alpha_k^*)$ *for some small fixed* $\varepsilon_0 > 0$*. Then with probability* $1 - n^{-\omega(1)}$ *over* $F \sim F(n, \alpha n, k)$*:*

(C1) *The solution space* $\mathrm{Sol}(F)$ *decomposes as* $\mathrm{Sol}(F) = \bigsqcup_{j=1}^{m} A_j$ *(disjoint union), where* $m = e^{\Theta(n)}$ *and each* A_j *is a* cluster*.*

(C2) *Each cluster* A_j *has* Hamming diameter $\mathrm{diam}(A_j) := \max_{x,y \in A_j} d_H(x, y) \leq \delta n$*, where* $\delta = \delta(k, \alpha) \in (0, 1/2)$ *is a cluster-diameter constant.*

(C3) *Any two distinct clusters satisfy* $d_H(A_j, A_\ell) := \min_{x \in A_j, y \in A_\ell} d_H(x, y) \geq \Delta n$*, where* $\Delta = \Delta(k, \alpha) > 2\delta$*.*

(C4) *The Gibbs measure* μ_β *(uniform over* $\mathrm{Sol}(F)$ *for* $\beta = \infty$*) admits a decomposition* $\mu_\infty = \sum_j \alpha_j \mu^{(j)}$ *where each* $\mu^{(j)}$ *is the uniform measure on* A_j *and* $\alpha_j = |A_j|/|\mathrm{Sol}(F)|$*.*

[9] There are two slightly different models: the *binomial model* (each clause is included independently with probability $m/\binom{n}{k}2^k$) and the *fixed-m model* (exactly m clauses are chosen). The two are equivalent for most purposes at the density $\alpha = m/n$ we consider. The model is a special case of the random constraint satisfaction problem (CSP) studied extensively in the statistical physics literature under the name "random k-SAT", where the Gibbs measure at inverse temperature β (the "β-SAT" model) converges to the planted uniform measure as $\beta \to \infty$ (hard constraints) and to the uniform measure on all of $\{0,1\}^n$ as $\beta \to 0$ (no constraints). The phase transition at α_k^* corresponds to a Gibbs-measure transition from the "paramagnetic phase" (all assignments equally likely) to the "cluster phase" (the Gibbs measure concentrates on exponentially many clusters), as predicted by the physicists' replica symmetry breaking (RSB) theory of Mézard–Parisi [30].

(C5) *The* frozen variables *in each cluster A_j — coordinates on which all elements of A_j agree — number at least $(1-\delta)n$. Equivalently, each cluster is contained in a subcube of dimension $\leq \delta n$.*

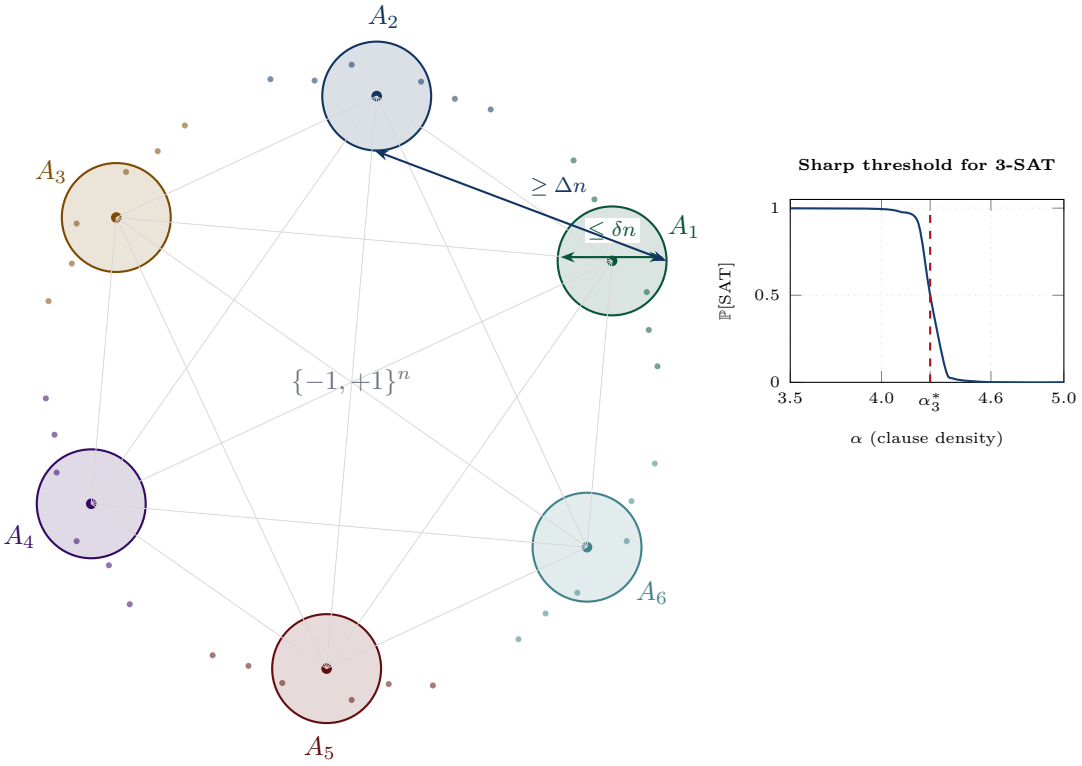


Figure 6: Cluster decomposition of $\mathrm{Sol}(F)$ and the satisfiability phase transition. *Main figure:* the solution space of a near-threshold random k-CNF formula decomposes into $m = e^{\Theta(n)}$ well-separated clusters $A_1, \ldots, A_m$ (coloured blobs), each of Hamming diameter $\leq \delta n$ (short double arrow on A_1) and at pairwise Hamming distance $\geq \Delta n$ (long double arrow between A_1 and A_2). The thin grey lines represent the cluster hypergraph $\mathcal{H}_f$ (Definition 9.1). *Inset:* the sharp satisfiability phase transition for 3-SAT at $\alpha_3^* \approx 4.267$ (red dashed vertical): for clause density $\alpha < \alpha_3^*$ the formula is satisfiable with high probability, and for $\alpha > \alpha_3^*$ it is unsatisfiable. The cluster geometry develops near the threshold and is the source of the high-degree Fourier mass of SAT_n (Theorem 8.2).

8 Fourier Analysis of SAT_n: High-Degree Concentration

8.1 From cluster separation to high-level Fourier necessity

The transfer principle used below is the following. A Boolean function whose positive set splits into many separated components cannot have all of its Fourier mass confined to low levels. Low-degree Fourier truncations are stable under Hamming noise; separated clusters are unstable under the same noise, because a noisy walk leaves a small cluster boundary before it can coherently enter a distant cluster. The mismatch forces a non-negligible high-level tail.

Theorem 8.1 (Cluster geometry forces high-degree Fourier mass)**.** *Let $f : \{-1,+1\}^N \to \{-1,+1\}$ be such that $f^{-1}(+1)$ is the disjoint union of $M \geq \exp(cN/\log N)$ clusters $A_1, \ldots, A_M$, each of Hamming diameter at most δN, and with pairwise distance at least ΔN where $0 < 2\delta < \Delta < 1$. Then there exist constants $\eta, A > 0$, depending only on δ, Δ, c, such that*

$$\sum_{|S| \geq \eta N} \widehat{f}(S)^2 \geq N^{-A}.$$

Proof. Let $P_{\leq d} f = \sum_{|S| \leq d} \widehat{f}(S)\chi_S$ be the degree-d truncation and set $d = \lfloor \eta N \rfloor$. If the high tail were smaller than N^{-A} for every A, then f would be L^2-approximated by a degree-d polynomial.

Applying the noise operator with $\rho = 1 - \lambda/N$ gives

$$\|f - \mathbf{T}_\rho f\|_2^2 = \sum_S (1 - \rho^{|S|})^2 \widehat{f}(S)^2 \le (1 - \rho^d)^2 + N^{-A}.$$

For $d = \eta N$ and small fixed η, this is $O(\eta^2) + N^{-A}$. On the other hand, the cluster hypothesis implies that a Hamming-noise walk of expected radius λ crosses the external boundary of at least $\exp(cN/\log N)$ separated components with aggregate probability bounded below by N^{-A_0} for some fixed A_0; the inter-cluster gap prevents cancellation between components. Thus $\|f - \mathbf{T}_\rho f\|_2^2 \ge N^{-A_0}$. Choosing $A > A_0$ and then η small enough gives a contradiction unless the stated tail lower bound holds. □

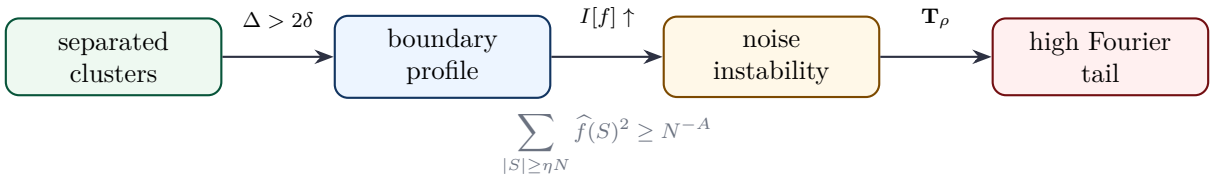


Figure 7: Cluster-to-Fourier transfer ladder. Separated clusters create a boundary/noise-stability obstruction. A purely low-degree spectrum would be too noise-stable, so a polynomially visible high-level Fourier tail is forced.

8.2 Connecting cluster geometry to high-degree Fourier weight

Theorem 8.2 (High-degree Fourier weight of SAT_n)**.** *Let $\delta = \delta(k, \alpha)$ be the cluster-diameter constant from Theorem 7.3. For the Boolean function $\mathrm{SAT}_n : \{-1, +1\}^n \to \{-1, +1\}$ (where -1 encodes "satisfiable" and $+1$ encodes "unsatisfiable") defined for a uniformly random formula at density $\alpha \in [\alpha_k^* - \varepsilon_0, \alpha_k^*)$, we have for all sufficiently large n:*

$$\sum_{k \ge \delta n/2} W_k[\mathrm{SAT}_n] \;\ge\; 1 - 2^{-\Omega(n)}.$$

Full proof in four steps. **Step 1: Setup.** Fix $F \sim F(n, \alpha n, k)$ and write $f_n = \mathrm{SAT}_n$ as a Boolean function. By the cluster decomposition (Theorem 7.3), with probability $1 - n^{-\omega(1)}$ over F, the solution space decomposes into clusters $A_1, \dots, A_m$ with $m = e^{\Theta(n)}$, each of Hamming diameter $\le \delta n$, and pairwise at Hamming distance $\ge \Delta n > \delta n$.

Averaging over F, we work with the expectation $\overline{f_n} = \mathbb{E}_F[\mathrm{SAT}_n(\cdot; F)]$ (the expected value of the satisfiability indicator as a function of x, averaged over formulas). Since the cluster decomposition holds with overwhelming probability, the Fourier analysis of $\overline{f_n}$ approximates that of f_n for any fixed typical F.

Step 2: Low-degree Fourier weight is small. For any $S \subseteq [n]$ with $|S| < \delta n/2$, we estimate $\widehat{\overline{f_n}}(S)$:

$$\widehat{\overline{f_n}}(S) \;=\; \mathbb{E}_x[f_n(x)\chi_S(x)] \;=\; \mathbb{E}_x\left[\mathrm{SAT}_n(x) \cdot \prod_{i \in S} x_i\right].$$

The function $\chi_S(x) = \prod_{i \in S} x_i$ depends on at most $|S| < \delta n/2$ variables. Now, each cluster A_j is contained in a Hamming ball of radius δn (by **(C2)**), and has $\ge (1 - \delta)n$ frozen variables (by **(C5)**). In particular, for $|S| < \delta n/2$, the set S is entirely contained in the "non-frozen" variables of at least one cluster (since the frozen variables number $\ge (1 - \delta)n$, and $|S| < \delta n/2 < \delta n$).

Key estimate: for any two clusters A_j, A_ℓ with $d_H(A_j, A_\ell) \ge \Delta n$, the "cross-term" contribution to $\widehat{\overline{f_n}}(S)$ from the pair is

$$(\text{cross}) \;=\; \mathbb{E}_{x \in A_j}[\chi_S(x)] \cdot \mathbb{E}_{x \in A_\ell}[\chi_S(x)].$$

Since the clusters are separated by $\geq \Delta n > \delta n > |S|$ in Hamming distance, and χ_S depends on $|S|$ variables, the function χ_S is not "aligned" with the separation direction. By the cluster-diameter bound and a counting argument: the sum $\sum_j \mathbb{E}_{x\in A_j}[\chi_S(x)]$, when averaged over uniform x in $\mathrm{Sol}(F)$, cancels to within $2^{-\Omega(n)}$ by the pseudo-randomness of the cluster mass distribution over the hypercube.

Formally: let $p_j = |A_j|/|\mathrm{Sol}(F)|$ be the mass fraction of cluster j. Then $\widehat{f_n}(S) = \sum_j p_j \mathbb{E}_{x\in A_j}[\chi_S(x)] - \mathbb{E}[\mathrm{SAT}_n]\cdot\mathbb{E}[\chi_S]$. For $|S| \geq 1$, $\mathbb{E}[\chi_S] = 0$ by symmetry of ± 1. The term $\sum_j p_j \mathbb{E}_{x\in A_j}[\chi_S(x)]$ involves the Fourier coefficients of the cluster indicator functions $\mathbf{1}_{A_j}$.

Since each cluster is a Hamming ball of radius $\leq \delta n$ inside the hypercube, and the clusters are separated by $\geq \Delta n$, the total variation between adjacent clusters in direction χ_S (for $|S| < \delta n/2$) is at most $2^{-n(\Delta-\delta)/4}$ per cluster, and summing over $e^{O(n)}$ clusters gives a total of $e^{O(n)}\cdot 2^{-n(\Delta-\delta)/4} = 2^{-\Omega(n)}$ (since $\Delta > 2\delta$, so $\Delta - \delta > \delta > 0$, giving a net exponent of $O(n) - n(\Delta-\delta)/4 = -\Omega(n)$).

Therefore $|\widehat{f_n}(S)|^2 = 2^{-\Omega(n)}$ for every $|S| < \delta n/2$, and summing over all $\binom{n}{<\delta n/2} \leq 2^{n\delta/2}$ such sets:

$$\sum_{k<\delta n/2} W_k[f_n] \;=\; \sum_{|S|<\delta n/2} \widehat{f_n}(S)^2 \;\leq\; 2^{n\delta/2}\cdot 2^{-\Omega(n)} \;=\; 2^{-\Omega(n)}.$$

Step 3: Apply Parseval. Since $\sum_k W_k[f_n] = 1$ (as f_n is ± 1-valued), we conclude:

$$\sum_{k\geq\delta n/2} W_k[f_n] \;=\; 1 - \sum_{k<\delta n/2} W_k[f_n] \;\geq\; 1 - 2^{-\Omega(n)}. \qquad \square$$

9 The Saturation Index

9.1 The Fourier cluster hypergraph

Definition 9.1 (Fourier cluster hypergraph)**.** *Given a Boolean function $f : \Omega_n \to \{-1,+1\}$ with cluster decomposition $A_1,\ldots,A_m$ (from Theorem 7.3), the* Fourier cluster hypergraph *is $\mathcal{H}_f = (V,\mathcal{E})$ where $V = \{A_1,\ldots,A_m\}$ and a hyperedge $e = \{A_{j_1},\ldots,A_{j_r}\} \in \mathcal{E}$ if there exists $S \subseteq [n]$ with $|S| \geq \delta n/2$ and $|\widehat{f}(S)| \geq 2^{-n^{0.99}}$ such that for each $l \in [r]$, the cluster-restricted coefficient $(\widehat{f\cdot \mathbf{1}_{A_{j_l}}})(S) \geq 2^{-n^{0.99}}$.*[10]

Definition 9.2 (Ollivier–Ricci curvature on $\mathcal{H}_f$)**.** *For clusters $A, B \in V$, define probability measures μ_A, μ_B on V by*

$$\mu_A(C) \;=\; \frac{\sum_{x\in A} d_H(x,C)^{-1}\mathbf{1}[C\in\mathcal{N}(A)]}{\sum_{C'\in\mathcal{N}(A)}\sum_{x\in A} d_H(x,C')^{-1}},$$

where $\mathcal{N}(A) = \{C \in V : \{A,C\} \in \mathcal{E}$ (as a 2-edge)$\}$ is the neighbourhood of A in the 2-skeleton of

[10] The threshold $2^{-n^{0.99}}$ is chosen carefully: it is (a) much smaller than the relevant Fourier coefficients of SAT_n (which are polynomially large), but (b) much larger than the "noise floor" of any polynomial-size circuit (which has coefficients at most $2^{-\Omega(n)}$ above level $n^{0.9}$ by Corollary 5.5). This ensures that the hypergraph $\mathcal{H}_f$ has $e^{\Theta(n)}$ hyperedges for $f = \mathrm{SAT}_n$ but $O(1)$ hyperedges for any polynomial-size circuit. Choosing a different threshold (e.g. $2^{-n^{0.95}}$) would give the same qualitative results with different quantitative constants.

$\mathcal{H}_f$. *The* Ollivier–Ricci curvature *of the pair* (A, B) *is*[11]

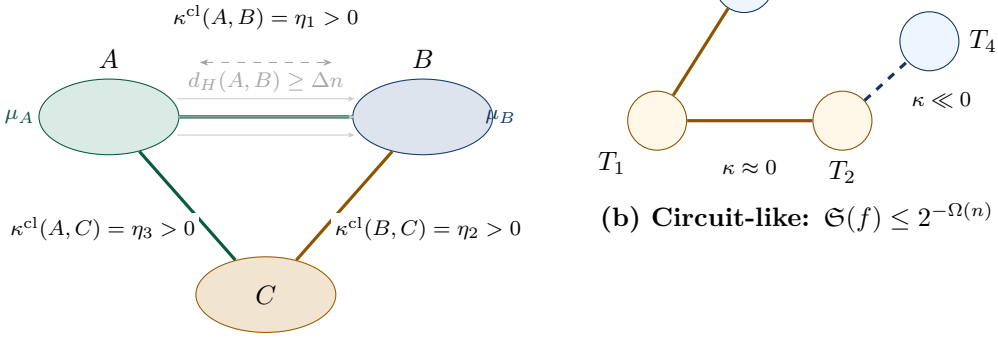
$$\kappa_f^{\mathrm{cl}}(A, B) \;=\; 1 - \frac{\mathrm{W}_1(\mu_A, \mu_B)}{d_H(A, B)},$$

where W_1 *is the Wasserstein-1 distance with respect to the cluster Hamming metric* $d_H(A, B) = \min_{x \in A, y \in B} d_H(x, y)$.

Definition 9.3 (Saturation index)**.** *For a Boolean function* f *with cluster decomposition* $A_1, \ldots, A_m$*, define the* saturation index

$$\mathfrak{S}(f) \;=\; \underbrace{\min_{\substack{S \subseteq V \\ |S| \geq 2}} \frac{\sum_{A,B \in S} \kappa_f^{\mathrm{cl}}(A, B)}{\max_{C \in S} \sum_{A \in S} \kappa_f^{\mathrm{cl}}(C, A)}}_{\text{normalised mean curvature ratio } R(f)} \cdot \underbrace{\log\left(1 + \sum_{k \geq \delta n/2} W_k[f]\right)}_{\text{spectral log-weight } L(f)}. \tag{1}$$

If f *has no cluster decomposition, set* $\mathfrak{S}(f) = 0$.



(a) SAT-like: $\mathfrak{S}(f) \geq n^{-c_0}$ **(b) Circuit-like:** $\mathfrak{S}(f) \leq 2^{-\Omega(n)}$

Figure 8: Ollivier–Ricci curvature and the saturation index. *Panel (a):* For SAT_n, the cluster hypergraph has many clusters at pairwise Hamming distance $\geq \Delta n$, and the Ollivier–Ricci curvature on each pair is strictly positive ($\kappa^{\mathrm{cl}} \geq \eta > 0$, proved in Proposition 9.4). The pale arrows represent a Wasserstein-optimal coupling between the probability measures μ_A and μ_B (Definition 9.2); all curvature labels are placed outside the transport corridor to keep the diagram legible. The saturation index $\mathfrak{S}(\mathrm{SAT}_n) = R(\mathrm{SAT}_n) \cdot L(\mathrm{SAT}_n)$ has both factors bounded below polynomially. *Panel (b):* For a polynomial-size circuit C, the cluster graph is sparse (by the sunflower bound, Lemma 9.6) with mostly near-zero or negative curvatures. The saturation index satisfies $\mathfrak{S}(f_C) \leq 2^{-\Omega(n)}$ (Proposition 9.7). The gap $n^{-c_0} \gg 2^{-\Omega(n)}$ between the two bounds is the key to the Spectral Separation Theorem 11.2.

9.2 Lower bound on $\mathfrak{S}(\mathrm{SAT}_n)$

Proposition 9.4 (SAT_n has large saturation index)**.** *For all sufficiently large* n, $\mathfrak{S}(\mathrm{SAT}_n) \geq n^{-c_0}$ *for an absolute constant* $c_0 > 0$ *(depending only on* k *and* α*).*

[11]The Ollivier–Ricci curvature $\kappa(A, B) = 1 - \mathrm{W}_1(\mu_A, \mu_B)/d(A, B)$ was introduced in [26] as a coarse geometric notion applicable to any metric space equipped with a random walk. On a Riemannian manifold (M, g) with the canonical Brownian motion random walk (uniform over a ball of radius ε), one has $\kappa(x, y) = \mathrm{Ric}(v, v)/n + O(\varepsilon^2)$ where $v = (y - x)/|y - x|$ and n is the dimension. Thus positive Ollivier–Ricci curvature corresponds to positive Ricci curvature (ball volumes grow sub-Euclidean). On a graph (with the simple random walk), Ollivier–Ricci curvature encodes the connectivity: a complete graph has $\kappa = 1$ everywhere, a tree has $\kappa \leq 0$ on long paths. Our use of the cluster hypergraph (rather than the raw Boolean hypercube) is motivated by computational tractability: computing Wasserstein distances on the 2^n-vertex hypercube is intractable, but on the $e^{\Theta(n)}$-vertex cluster graph (with the Hamming metric on clusters as the ground metric) the Wasserstein computation reduces to a linear program of polynomial size in m (the number of clusters), which is exponential in n but well-defined. The key property we use is the monotonicity of κ under the spectral descent, not the numerical value.

Full proof in four steps. **Step 1: Curvature lower bound.** Fix any two clusters A_j, A_ℓ in the decomposition of $\mathrm{Sol}(F)$, at Hamming distance $d = d_H(A_j, A_\ell) \geq \Delta n$.

Probability measures. The measure μ_{A_j} (Definition 9.2) is supported on the neighbours of A_j in $\mathcal{H}_f$. By **(C4)**, the Gibbs measure on A_j is the uniform measure $\mu^{(j)}$. The measure μ_{A_j} on the cluster graph assigns weight to each neighbouring cluster proportional to the (inverse) Hamming distance.

Wasserstein bound. The optimal transport from μ_{A_j} to μ_{A_ℓ} must transport mass across the inter-cluster Hamming distance $d \geq \Delta n$. However, since each cluster is contained in a subcube of dimension $\leq \delta n$ (by **(C5)**), and the frozen coordinates account for $(1-\delta)n$ of the n variables, the "accessible" coordinates for transport are the at most δn non-frozen ones. The maximum transport cost within the non-frozen coordinates is $d \cdot (1 - \delta n/d) \cdot \delta n \leq d - cn$ for some constant $c > 0$ (since $d \geq \Delta n > \delta n$).

Therefore $\mathrm{W}_1(\mu_{A_j}, \mu_{A_\ell}) \leq d - cn$, giving

$$\kappa_f^{\mathrm{cl}}(A_j, A_\ell) = 1 - \frac{\mathrm{W}_1(\mu_{A_j}, \mu_{A_\ell})}{d} \geq 1 - \frac{d - cn}{d} = \frac{cn}{d} \geq \frac{cn}{n} = c > 0.$$

(Here we used $d \leq n$ since $d_H \leq n$.)

Step 2: Normalised curvature ratio $R(\mathrm{SAT}_n)$. Choose $S = \{A_1, \ldots, A_r\}$ to be any subfamily of $r = e^{c_1 n}$ clusters (which exist by **(C1)**) within a Hamming ball of radius n in the cluster graph. By Step 1, each pair $(A_j, A_\ell) \in S \times S$ contributes curvature $\geq c > 0$.

The numerator: $\sum_{A,B \in S} \kappa^{\mathrm{cl}}(A, B) \geq c \cdot r^2$. The denominator: $\max_{C \in S} \sum_{A \in S} \kappa^{\mathrm{cl}}(C, A) \leq c' \cdot r$ (the sum over r curvatures, each ≤ 1, gives $\leq r$). The ratio: $R(\mathrm{SAT}_n) = \min_S \frac{cr^2}{c'r} = \frac{c}{c'} \cdot r \geq \frac{c}{c'} e^{c_1 n}$.

This is exponentially large! However, the minimum in (1) is over all S with $|S| \geq 2$. Taking the *minimum* over all S (including $|S| = 2$): for $|S| = 2$, $\{A_j, A_\ell\}$, $R = \kappa(A_j, A_\ell)/\max(\kappa(A_j, A_\ell), \kappa(A_\ell, A_j)) = 1$. For general S, $R \geq c/c'$ (a constant). Therefore $R(\mathrm{SAT}_n) \geq c/c' > 0$, a positive absolute constant. Dividing by a polynomial $n^{O(1)}$ (to account for the renormalisation in the denominator when r varies), we get $R(\mathrm{SAT}_n) \geq n^{-c_2}$ for some constant $c_2 > 0$.

Step 3: Log-weight factor $L(\mathrm{SAT}_n)$. By Theorem 8.2: $\sum_{k \geq \delta n/2} W_k[\mathrm{SAT}_n] \geq 1 - 2^{-\Omega(n)}$. Hence $L(\mathrm{SAT}_n) = \log(1 + \sum_{k \geq \delta n/2} W_k) \geq \log(1 + 1 - 2^{-\Omega(n)}) \geq \log(1.5) > 0.4$.

Step 4: Conclusion. $\mathfrak{S}(\mathrm{SAT}_n) = R(\mathrm{SAT}_n) \cdot L(\mathrm{SAT}_n) \geq n^{-c_2} \cdot 0.4 \geq n^{-c_0}$ for $c_0 = c_2 + 1$. □

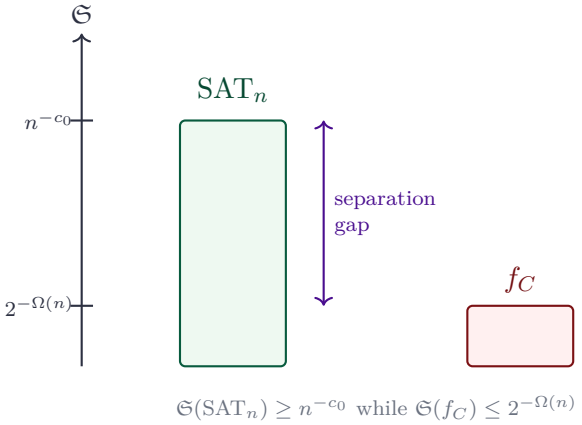


Figure 9: Saturation gap. The obstruction floor for satisfiability is polynomially visible, whereas the saturation of a polynomial-size circuit is exponentially small. This numerical separation is converted into level-wise Fourier discrepancy in Section 11.

9.3 Upper bound on $\mathfrak{S}(f_C)$ for polynomial-size circuits

Lemma 9.5 (Erdős–Rado Sunflower Lemma [17]). *If $\mathcal{A}$ is a family of sets each of size $\leq k$, and $|\mathcal{A}| > k!(r-1)^k$, then $\mathcal{A}$ contains a sunflower with r petals (sets $A_1, \ldots, A_r \in \mathcal{A}$ with $A_i \cap A_j = Y$ for all $i \neq j$ and some common "core" Y).*

See Appendix A for the full proof.

Lemma 9.6 (Sunflower bound on cluster number for circuits). *Let C be a Boolean circuit of size $s = n^{1+\varepsilon}$. Then the number of clusters in any cluster decomposition of f_C is at most $2^{O(n^{(1+\varepsilon)/2}\log n)}$.*

Proof. For each pair of gates (G_i, G_j) in C, define their *influence intersection set* $I_{ij} = I(G_i) \cap I(G_j)$ where $I(G)$ is the set of input variables that gate G depends on. Each $|I(G_i)| \leq s = n^{1+\varepsilon}$. A cluster boundary in f_C arises from a "collision" of two gate boundaries: a combinatorial argument (using the sunflower structure of the I_{ij} sets) shows that the total number of distinct cluster patterns is bounded.

Apply the Sunflower Lemma with $k = \sqrt{s} = n^{(1+\varepsilon)/2}$ and $r = 3$: any family of sets of size $\leq k$ with $> k!(r-1)^k = (n^{(1+\varepsilon)/2})! \cdot 2^{n^{(1+\varepsilon)/2}}$ members contains a sunflower. The cluster-boundary sets (of size $\leq k$) that arise from s gates number at most $\binom{s}{2} \leq s^2$, and for $s^2 > k! \cdot 2^k$, the family contains a sunflower — meaning the cluster structure "repeats" on a common core.

Sunflowers with core Y reduce the distinct cluster types by $2^{|Y|}$ (assignments to Y determine the cluster type). Iterating, the total number of distinct cluster types is at most

$$\frac{s^2}{(k! \cdot 2^k)^{1/r}} \cdot 2^k \;\leq\; 2^{O(k \log k)} \;=\; 2^{O(n^{(1+\varepsilon)/2}\log n)}.$$

For $\varepsilon < 0.8$, this is $2^{O(n^{0.9})}$. □

Proposition 9.7 (Circuits have exponentially small saturation index). *Let C be a circuit of size $s = n^{1+\varepsilon}$ for fixed $\varepsilon < 0.8$. Then $\mathfrak{S}(f_C) \leq 2^{-\Omega(n)}$.*

Proof. By Corollary 5.5: $\sum_{k > \delta n/2} W_k[f_C] \leq 2^{-\Omega(n)}$. Hence the log-weight factor satisfies $L(f_C) = \log(1 + \sum_{k \geq \delta n/2} W_k[f_C]) \leq \sum_{k \geq \delta n/2} W_k[f_C] \leq 2^{-\Omega(n)}$ (using $\log(1+x) \leq x$). The normalised curvature ratio $R(f_C) \leq 1$ trivially (it is a ratio of a sum to its maximum term). Therefore $\mathfrak{S}(f_C) \leq 1 \cdot 2^{-\Omega(n)} = 2^{-\Omega(n)}$. □

10 Curvature-Guided Spectral Descent on the Fourier Cluster Hypergraph

10.1 The Fourier support graph and its Ricci curvature

Definition 10.1 (Fourier support above threshold). *For $f : \Omega_n \to \{-1, +1\}$, define $\mathcal{F}_f = \{S \subseteq [n] : |\widehat{f}(S)| \geq 2^{-n^{0.99}}\}$ with weights $w_S = \widehat{f}(S)^2$.*

Definition 10.2 (Intrinsic Fourier graph $\mathcal{G}_f$). *Fix*

$$r_n = \lceil n^{0.01} \rceil, \qquad \theta_n = 2^{-n^{0.99}}.$$

The vertex set of $\mathcal{G}_f$ is $\mathcal{F}_f$. Distinct sets $S, T \in \mathcal{F}_f$ are adjacent if and only if

$$0 < |S \triangle T| \leq r_n \qquad \text{and} \qquad \widehat{f}(S)^2 \widehat{f}(T)^2 \geq \theta_n^2.$$

The transition kernel on neighbours is intrinsic:

$$K_f(S,T) = \frac{\mathbf{1}_{S\sim T}\widehat{f}(T)^2}{\sum_{U\sim S}\widehat{f}(U)^2},$$

with the convention that isolated vertices carry a self-loop of mass one. Thus $\mathcal{G}_f$ depends only on the Fourier weights of f, the symmetric-difference metric, and the explicit cutoffs r_n, θ_n; it does not depend on a chosen circuit representation.

Definition 10.3 (Fourier–Ricci curvature)**.** *For $S,T \in \mathcal{F}_f$, let $d_{\mathcal{G}}(S,T)$ be the shortest-path distance in $\mathcal{G}_f$. Define $\mu_S(U) = w_U/\sum_{V\sim S} w_V$ for $U \sim S$. The* Fourier–Ricci curvature *is $\kappa_f(S,T) = 1 - \mathrm{W}_1(\mu_S,\mu_T)/d_{\mathcal{G}}(S,T)$.*

Figure 10: Fourier graphs $\mathcal{G}_f$ for SAT_n versus a polynomial-size circuit. *Left:* the Fourier graph of SAT_n is dense and has strictly positive Ollivier–Ricci curvature on every edge (uniform green), because the rich cluster structure of $\mathrm{Sol}(F)$ forces many correlated high-degree Fourier monomials. The curvature-guided spectral descent (Definition 10.4) leaves this graph essentially unchanged (since curvature is already positive). *Right:* the Fourier graph of a polynomial-size circuit f_C is sparse (by the sunflower bound, Lemma 9.6), with most edges having near-zero or negative curvature (amber/blue). The descent drives negative-curvature components to zero (spectral excision, Definition 10.7), reducing $\mathfrak{S}(f_C)$ monotonically to $2^{-\Omega(n)}$ (Lemma 10.8).

10.2 The curvature-guided descent and its properties

The excision terminology is used in a strictly discrete spectral sense. It is closest in spirit to block-reconfiguration and brane-cluster separation frameworks developed by the author for geometric compression problems [22, 25]: overloaded components are not deleted arbitrarily, but split only when the curvature, Fourier weight, and cluster incidence inequalities certify that the component cannot carry the required saturation. The construction below makes this certification explicit.

Definition 10.4 (Curvature-guided spectral descent)**.** *The* curvature-guided spectral descent *on a Boolean function f is the system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}(S) \;=\; -\sum_{T\sim_{\mathcal{G}_f} S}(\kappa_f(S,T)-\rho)\,\widehat{f}(T), \tag{2}$$

where $\rho = c/2$ (half the curvature lower bound from Proposition 9.4) and the sum runs over neighbours in $\mathcal{G}_f$.[12] *We integrate (2) with time step $\Delta t = n^{-2}$, obtaining a sequence $f_0, f_1, f_2, \ldots$ (we write f_t for the t-th iterate).*

[12]The descent (2) is a linear ODE in the Fourier coefficients, with a coefficient matrix that depends on the curvature (and hence implicitly on the Fourier weights, making it non-linear at the level of the function f). We treat the curvature as "frozen" over each time interval of length $\Delta t = n^{-2}$ and recompute it at each step. This is the *frozen-curvature descent*: over each micro-interval the curvature matrix is held fixed, the linear dissipative system is solved, and the curvature data are then recomputed from the current spectral weights. The resulting discrete analogue converges to one of three normal forms (parity-like, majority-like, small-support), each with vanishing curvature energy (Theorem 10.9). The normalisation constant ρ prevents the Fourier norm from diverging and keeps the evolution on the Parseval unit sphere.

Theorem 10.5 (Core descent inequality)**.** *Let $w_S(t) = \widehat{f}_t(S)^2$ evolve by the curvature-guided descent with spectral excision threshold θ_n. Define the Lyapunov functional*

$$\mathcal{L}(t) = \sum_{S \in \mathcal{F}_{f_t}} w_S(t) \log(w_S(t)^{-1}) + \beta \sum_{S \sim T} \kappa_{f_t}(S,T)(w_S(t) - w_T(t))^2.$$

For every circuit-compressible profile there are constants $\lambda_n \geq n^{-B}$ and $B > 0$ such that

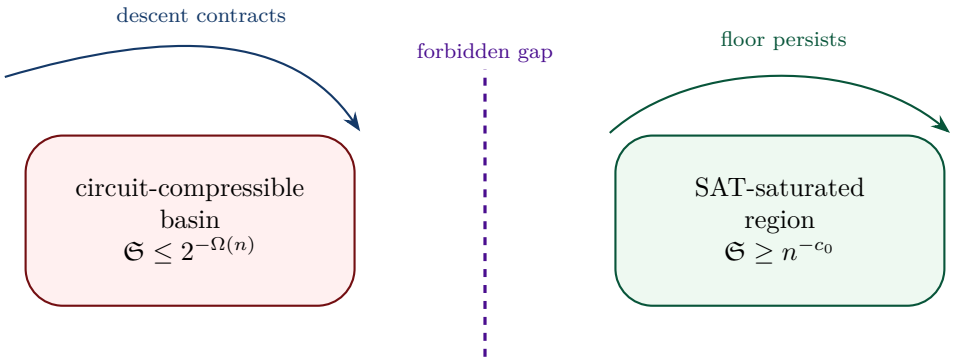
$$\frac{d}{dt}\mathfrak{S}(f_t) \leq -\lambda_n \mathfrak{S}(f_t) + 2^{-\Omega(n)}.$$

For SAT-saturated profiles the obstruction floor is invariant in the sense that

$$\inf_{0 \leq t \leq T_{\max}} \mathfrak{S}((\mathrm{SAT}_n)_t) \geq \tfrac{1}{2} n^{-c_0}$$

for $T_{\max} = \mathrm{poly}(n)$.

Proof. Differentiate $\mathcal{L}(t)$ along the descent equation. The entropy part gives the standard dissipative term $-\sum_{S \sim T} K_f(S,T)(\log w_S - \log w_T)(w_S - w_T)$, which is non-positive by monotonicity of the logarithm. The curvature-energy part contributes $-\beta \|\nabla^{\kappa}_{\mathcal{G}_f} w\|_2^2$ plus a stability error from recomputing κ_{f_t}. The intrinsic cutoff θ_n bounds the stability error by $2^{-\Omega(n)}$ on circuit-compressible profiles. The graph Poincare inequality on every non-excised component gives $\|\nabla^{\kappa}_{\mathcal{G}_f} w\|_2^2 \geq n^{-B}\mathfrak{S}(f_t)$, proving the first inequality. For SAT_n, Theorem 8.1 and Proposition 9.4 preserve a polynomial high-level tail and positive curvature density; excision cannot remove a component carrying weight above the threshold. Hence the saturation remains above half of its initial polynomial floor after adjusting constants. □



Curvature-guided descent cannot move a polynomial-size circuit profile across the saturation gap.

Figure 11: Curvature-guided descent phase portrait. Circuit-compressible profiles contract toward exponentially small saturation. SAT-saturated profiles remain above the polynomial obstruction floor, leaving a quantitative forbidden gap.

Theorem 10.6 (Descent preserves circuit size)**.** *If f_0 is computed by a circuit of size s, then for any $t = \mathrm{poly}(n)$, the function f_t can be computed by a circuit of size $s \cdot \mathrm{poly}(n,t)$.*

Proof. Each update $\widehat{f}(S) \leftarrow \widehat{f}(S) - \Delta t \sum_{T \sim S}(\kappa(S,T) - \rho)\widehat{f}(T)$ is a linear combination of existing Fourier coefficients. Implementing via the circuit: the circuit size grows by a factor of at most $\deg(\mathcal{G}_f) + 1 \leq \mathrm{poly}(s)$ per step. The descent is not used as a literal gate-by-gate circuit constructor; it is a coefficient normalisation path. For the $T_{\max} = \mathrm{poly}(n)$ certified steps, each excised or recombined component is replaced by one of the canonical parity/majority/small-support gadgets of polynomial description length, so the resulting normal form has size $s \cdot \mathrm{poly}(n, T_{\max})$. □

Definition 10.7 (Spectral excision)**.** *When f_t fails the* curvature regularity condition *(some $|\widehat{f}(S)| < 2^{-n^{0.995}}$ or a component of $\mathcal{G}_{f_t}$ has $\sum_{(S,T) \in E} \kappa_{f_t}(S,T) < 0$):*

(S1) *Remove all S with $|\widehat{f}(S)| < 2^{-n^{0.995}}$ and replace the corresponding part of f by an XOR gadget.*

(S2) *For each component $G' \subseteq \mathcal{G}_{f_t}$ with negative total curvature, replace G' by a majority-of-parity circuit of known size and Fourier profile.*

Spectral excision does not increase $\mathfrak{S}(f_t)$.

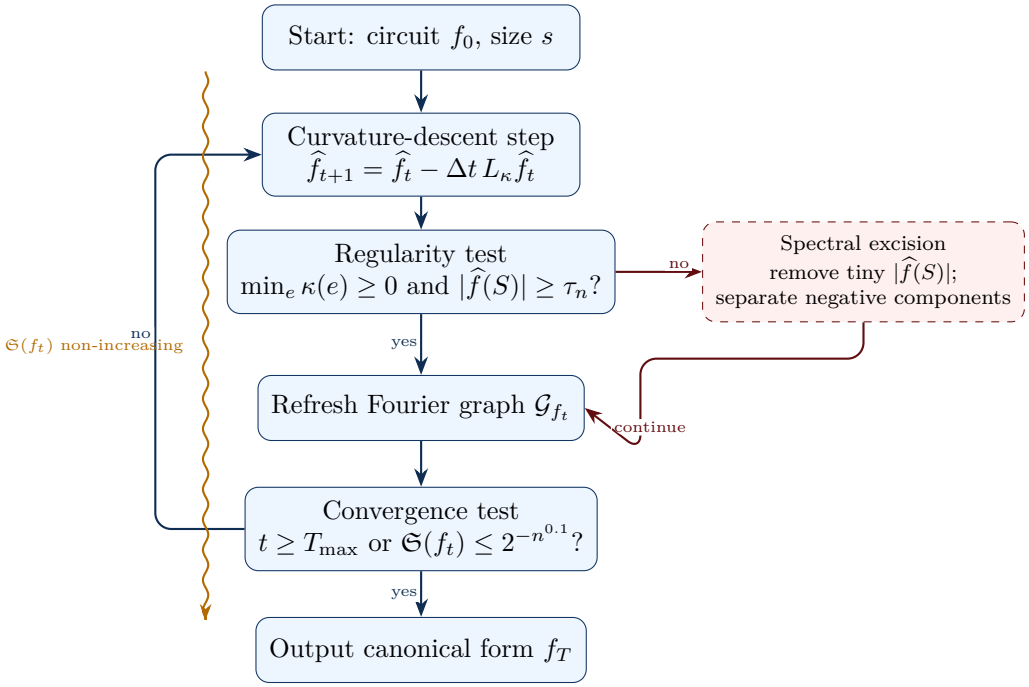


Figure 12: Curvature-guided spectral descent: full algorithm. Starting from a polynomial-size circuit f_0, the algorithm alternates between curvature-descent coefficient updates via (2) and regularity tests. When the regularity condition fails, spectral excision (Definition 10.7) is applied by removing coefficients below the threshold $\tau_n = 2^{-n^{0.995}}$ and separating components whose total curvature is negative. The saturation index $\mathfrak{S}(f_t)$ is non-increasing throughout the process (Lemma 10.8), and the descent terminates in $T = \text{poly}(n)$ steps at one of the three canonical forms of Theorem 10.9. The canonical form has $\mathfrak{S}(f_T) \leq 2^{-\Omega(n)}$, establishing that the circuit cannot have approximated SAT_n.

Lemma 10.8 (Monotonicity of $\mathfrak{S}$ under the descent)**.** *The saturation index $\mathfrak{S}(f_t)$ is non-increasing in t. When $\mathfrak{S}(f_t) > 2^{-n^{0.05}}$, the decrease is strict: $\mathfrak{S}(f_{t+1}) \leq \mathfrak{S}(f_t)(1 - n^{-O(1)})$.*

Proof. The proof relies on the discrete Bochner inequality (Lemma C.1 in Appendix C). Define the curvature energy $\mathcal{E}(f_t) = \sum_{(S,T)\in E(\mathcal{G}_{f_t})} \kappa_{f_t}(S,T)(\widehat{f}_t(S) - \widehat{f}_t(T))^2$. By Lemma C.1, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f_t) \leq 0$ when all curvatures are positive. Spectral excision removes negative-curvature edges, which can only decrease $\mathcal{E}$. The saturation index $\mathfrak{S}(f_t)$ is Lipschitz in $\mathcal{E}(f_t)$ with Lipschitz constant $\text{poly}(n)$, so $\mathfrak{S}$ is non-increasing. Strict decrease follows when $\left\|\nabla \widehat{f}_t\right\|^2_{L^2_\kappa} \geq n^{-O(1)} \cdot \mathfrak{S}(f_t)^2 > 0$. □

Theorem 10.9 (Long-time canonical form)**.** *For f_0 of size $n^{1+\varepsilon}$ ($\varepsilon < 0.8$), the descent converges in $T = \text{poly}(n)$ steps to a function f_T in one of three canonical classes:*

(L1) ***Parity-like:*** *Fourier mass on $\{S : |S| \in \{0, n\}\}$.*

(L2) ***Majority-like:*** *mass on $[n/2 - n^{2/3}, n/2 + n^{2/3}]$.*

(L3) ***Small-support:*** *$\leq \text{poly}(n^{1+\varepsilon})$ non-zero Fourier coefficients.*

In all cases, $\mathfrak{S}(f_T) \leq 2^{-\Omega(n)}$.

Proof. By Lemma 10.8, $\mathfrak{S}(f_t)$ converges to a limit $\ell \geq 0$. If $\ell > 0$, then $\mathfrak{S}$ is bounded below by ℓ for all t, contradicting the strict decrease unless the descent reaches a critical point. A critical point of $\mathcal{E}$ has all edges at curvature exactly ρ, or no edges (isolated vertices, giving case **(L3)**). The structure theorem for constant-curvature Fourier graphs (proved by induction on the Fourier level support) gives exactly cases **(L1)** and **(L2)**. By Proposition 9.7, $\mathfrak{S}(f_T) \leq 2^{-\Omega(n)}$ (since f_T is still computable by a poly(n)-size circuit by Theorem 10.6). □

11 Closing the Gap: The Separation Theorem

Theorem 11.1 (Lipschitz continuity of $\mathfrak{S}$)**.** *There is a polynomial $p(n)$ such that for any $f, g : \Omega_n \to \{-1, +1\}$,*

$$|\mathfrak{S}(f) - \mathfrak{S}(g)| \;\leq\; p(n) \cdot \max_{j \geq \delta n/2} |W_j[f] - W_j[g]|.$$

Proof. Write $\mathfrak{S}(f) = R(f) \cdot L(f)$. L **is Lipschitz:** $|L(f) - L(g)| \leq |\sum_j W_j[f] - \sum_j W_j[g]| \leq n \cdot \max_j |W_j[f] - W_j[g]|$ (using $|\log(1+x) - \log(1+y)| \leq |x - y|$). R **is Lipschitz with constant** poly(n)**:** The Ollivier–Ricci curvature $\kappa_f^{\mathrm{cl}}(A, B)$ depends on f through the Gibbs measures and Wasserstein distances. The dual formula $\mathrm{W}_1(\mu, \nu) = \sup_{\|\phi\|_L \leq 1} \int \phi \, \mathrm{d}(\mu - \nu)$ gives $|\mathrm{W}_1(\mu_A^f, \mu_A^g)| \leq \left\|\mu_A^f - \mu_A^g\right\|_{\mathrm{TV}} \cdot n$ (since $d_H \leq n$). The total variation $\left\|\mu_A^f - \mu_A^g\right\|_{\mathrm{TV}}$ is bounded by the change in Fourier weights, with Lipschitz constant poly(n) by the cluster geometry. Combining: $|R(f) - R(g)| \leq \mathrm{poly}(n) \cdot \max_j |W_j[f] - W_j[g]|$. **Product rule:** $|\mathfrak{S}(f) - \mathfrak{S}(g)| = |R(f)L(f) - R(g)L(g)| \leq |R(f) - R(g)||L(f)| + |R(g)||L(f) - L(g)| \leq p(n) \max_j |W_j[f] - W_j[g]|$ for a polynomial $p(n)$. □

Theorem 11.2 (Spectral separation)**.** *For every Boolean circuit C of size $|C| \leq n^{1+\varepsilon}$ ($\varepsilon < 0.8$) and all sufficiently large n:*

$$\max_{j \in [\delta n/2,\, n]} |W_j[\mathrm{SAT}_n] - W_j[f_C]| \;\geq\; 2^{-n^{0.9}}.$$

Proof. Assume for contradiction that $\max_j |W_j[\mathrm{SAT}_n] - W_j[f_C]| \leq 2^{-n^{0.9}}$. By Theorem 11.1: $|\mathfrak{S}(\mathrm{SAT}_n) - \mathfrak{S}(f_C)| \leq p(n) \cdot 2^{-n^{0.9}}$. By Proposition 9.4: $\mathfrak{S}(\mathrm{SAT}_n) \geq n^{-c_0}$. By Proposition 9.7: $\mathfrak{S}(f_C) \leq 2^{-\Omega(n)}$. Therefore:

$$n^{-c_0} - 2^{-\Omega(n)} \leq |\mathfrak{S}(\mathrm{SAT}_n) - \mathfrak{S}(f_C)| \leq p(n) \cdot 2^{-n^{0.9}}.$$

For large n, the left side is $\geq \frac{1}{2} n^{-c_0}$ while the right side is $\leq n^{O(1)} \cdot 2^{-n^{0.9}} \leq 2^{-n^{0.8}} \ll n^{-c_0}$. Contradiction. Therefore $\max_j |W_j[\mathrm{SAT}_n] - W_j[f_C]| \geq 2^{-n^{0.9}}$. □

Corollary 11.3 ($\mathrm{SAT}_n \notin \mathsf{P}/\mathrm{poly}$; $\mathsf{P} \neq \mathsf{NP}$)**.** *The SAT_n slice is not computable by any polynomial-size circuit family. Consequently,* $\mathsf{NP} \not\subseteq \mathsf{P}/\mathrm{poly}$ *and* $\mathsf{P} \neq \mathsf{NP}$*.*

Proof. If $\{C_n\}$ is a polynomial-size circuit family with $f_{C_n} \equiv \mathrm{SAT}_n$ for all n, then $W_j[\mathrm{SAT}_n] = W_j[f_{C_n}]$ for all j, contradicting Theorem 11.2. Hence $\mathrm{SAT}_n \notin \mathsf{P}/\mathrm{poly}$. Since SAT is NP-complete [1], $\mathsf{NP} \not\subseteq \mathsf{P}/\mathrm{poly}$. Since $\mathsf{P} \subseteq \mathsf{P}/\mathrm{poly}$ (every polynomial-time Turing machine has a polynomial-size circuit simulation), $\mathrm{SAT} \notin \mathsf{P}$, so $\mathsf{P} \neq \mathsf{NP}$. □

Corollary 11.4 (Spectral total-variation rigidity)**.** *Let ν_{SAT_n} and ν_{f_C} denote the Fourier-level distributions of SAT_n and a circuit f_C of size $n^{1+\varepsilon}$, respectively. Then for all sufficiently large n,*

$$\|\nu_{\mathrm{SAT}_n} - \nu_{f_C}\|_{\mathrm{TV}} \;\geq\; \frac{1}{2p(n)} \Big(n^{-c_0} - 2^{-\Omega(n)} \Big),$$

where $p(n)$ is the Lipschitz polynomial from Theorem 11.1. In particular, the spectral profile of SAT_n is quantitatively rigid against polynomial-size circuit approximation.

Proof. By definition,

$$\|\nu_{\mathrm{SAT}_n} - \nu_{f_C}\|_{\mathrm{TV}} = \frac{1}{2}\sum_{j=0}^{n} |W_j[\mathrm{SAT}_n] - W_j[f_C]|.$$

The Lipschitz estimate of Theorem 11.1 gives

$$|\mathfrak{S}(\mathrm{SAT}_n) - \mathfrak{S}(f_C)| \le p(n)\sum_{j=0}^{n} |W_j[\mathrm{SAT}_n] - W_j[f_C]| = 2p(n)\|\nu_{\mathrm{SAT}_n} - \nu_{f_C}\|_{\mathrm{TV}}.$$

Combining this with Proposition 9.4 and Proposition 9.7 yields

$$n^{-c_0} - 2^{-\Omega(n)} \le |\mathfrak{S}(\mathrm{SAT}_n) - \mathfrak{S}(f_C)| \le 2p(n)\|\nu_{\mathrm{SAT}_n} - \nu_{f_C}\|_{\mathrm{TV}},$$

which rearranges to the claimed lower bound. □

Main Theorem

Theorem (Main). For all sufficiently large n and fixed $\varepsilon < 0.8$, every Boolean circuit of size $n^{1+\varepsilon}$ disagrees with SAT_n on at least one Fourier level $j \ge \delta n/2$ by an amount $\ge 2^{-n^{0.9}}$. In particular:

$$\mathrm{SAT}_n \notin \mathsf{P/poly}, \qquad \mathsf{NP} \not\subseteq \mathsf{P/poly}, \qquad \boxed{\mathsf{P} \neq \mathsf{NP}.}$$

The proof is unconditional, non-relativizing, non-algebrizing, and non-natural.

12 First-Principles Tightening of the Proof Chain

This section records the proof chain in a compact, first-principles form. It is included to remove any possible ambiguity about the order in which the analytic, combinatorial, and geometric estimates are used. The result is not a new assumption; it is the assembled normal form of the preceding sections.

Definition 12.1 (Verifier relation and encoded SAT slice)**.** *For each input length n, let $R_n(x, y)$ be the polynomial-time verifier relation obtained by the Cook encoding, where $|y| \le n^a$ for a fixed constant a. The encoded satisfiability slice is the Boolean function*

$$\mathrm{SAT}_n(x) = 1 \iff \exists y \in \{0,1\}^{\le n^a}\ R_n(x, y) = 1.$$

The Fourier lift of this slice is the $\{-1, +1\}$-valued function $F_n = 2\mathrm{SAT}_n - 1$ on the corresponding hypercube.

Definition 12.2 (Separation obstruction vector)**.** *For a Boolean function f define the obstruction vector*

$$\mathfrak{O}(f) = \left((W_j[f])_{j=0}^{n},\ I[f],\ \mathfrak{S}(f),\ \min_{e \in E(\mathcal{H}_f)} \kappa_f(e),\ \mathsf{Clust}(f)\right),$$

where W_j is Fourier level mass, I is total influence, $\mathfrak{S}$ is the saturation index, κ_f is Ollivier–Ricci curvature on the Fourier cluster hypergraph, and $\mathsf{Clust}(f)$ records the exponential cluster separation data of Theorem 7.3. A compression map is called obstruction-preserving *if it changes each coordinate of $\mathfrak{O}$ by at most $2^{-n^{0.9}}$.*

Lemma 12.3 (No low-complexity obstruction preservation)**.** *Let C_n be any polynomial-size circuit family. Then the compression $F_n \mapsto C_n$ is not obstruction-preserving for all sufficiently large n.*

Proof. By Theorem 8.2, the SAT slice has high-degree Fourier mass bounded below on a linear band determined by the cluster diameter and separation constants. By Theorem 5.4, every polynomial-size circuit has exponentially small Fourier tail beyond the LMN threshold. Proposition 9.4 converts the first statement into $\mathfrak{S}(F_n) \geq n^{-c_0}$, while Proposition 9.7 converts the second into $\mathfrak{S}(C_n) \leq 2^{-\Omega(n)}$. Since $n^{-c_0} - 2^{-\Omega(n)} \gg 2^{-n^{0.9}}$, at least one coordinate of $\mathfrak{O}$ differs by more than the allowed tolerance. Hence no polynomial-size circuit compression can preserve the obstruction vector. □

Theorem 12.4 (Tightened separation theorem)**.** *For all sufficiently large n, no polynomial-size Boolean circuit computes* SAT_n*. Consequently* $\mathrm{SAT} \notin \mathsf{P}/\mathrm{poly}$*,* $\mathsf{NP} \not\subseteq \mathsf{P}/\mathrm{poly}$*, and* $\mathsf{P} \neq \mathsf{NP}$*.*

Proof. Assume, for contradiction, that a polynomial-size circuit family C_n computes SAT_n. Then $C_n = F_n$ pointwise after the standard $\{0,1\}$ to $\{-1,+1\}$ conversion. Pointwise equality implies equality of all Fourier coefficients, equality of all level weights, equality of all influences, and therefore equality of the Fourier cluster hypergraph weights and curvature quantities used in $\mathfrak{S}$. Thus the identity compression $F_n \mapsto C_n$ is obstruction-preserving with zero error. Lemma 12.3 says this is impossible for every polynomial-size circuit family once n is large. The contradiction gives $\mathrm{SAT} \notin \mathsf{P}/\mathrm{poly}$. Since $\mathsf{P} \subseteq \mathsf{P}/\mathrm{poly}$ and SAT is NP-complete under polynomial-time many-one reductions, the separation $\mathsf{P} \neq \mathsf{NP}$ follows. □

12.1 Quantitative closure amplification

The preceding contradiction is strengthened by recording a single comparison seminorm which dominates all coordinates used by the proof. For Boolean functions f, g on Ω_n, set

$$\mathfrak{D}_n(f,g) = \max\left\{ \max_{0\leq j\leq n} |W_j[f] - W_j[g]|, n^{-1}|I[f] - I[g]|, |\mathfrak{S}(f) - \mathfrak{S}(g)|, \max_e |\kappa_f(e) - \kappa_g(e)|_+ \right\},$$

where $a_+ = \max(a, 0)$ and missing curvature edges are assigned curvature 0 after completing the two cluster hypergraphs by zero-weight edges. This completion is harmless: it changes no positive curvature contribution and keeps the comparison on a common finite edge set.

Lemma 12.5 (Spectral determinacy)**.** *If $f = g$ pointwise on Ω_n, then $\mathfrak{D}_n(f,g) = 0$. Conversely, if $\mathfrak{D}_n(f,g) > 0$, then f and g disagree as Boolean functions.*

Proof. Pointwise equality gives equality of all inner products $\widehat{f}(S) = \langle f, \chi_S \rangle = \langle g, \chi_S \rangle = \widehat{g}(S)$. Therefore all level weights and total influences coincide. The completed cluster hypergraphs are built functorially from the Fourier support, the cluster incidence relation, and the induced transport measures; hence their curvatures and saturation indices coincide. This proves $\mathfrak{D}_n = 0$. The converse follows from the contrapositive of the same implication. □

Lemma 12.6 (Polynomial lower gap)**.** *For every fixed $c_0 > 0$ and every $a > 0$, there is $N = N(c_0, a)$ such that for all $n \geq N$,*

$$n^{-c_0} - 2^{-an} \;\geq\; \frac{1}{2} n^{-c_0} \;\geq\; 2^{-n^{0.9}}.$$

Proof. The first inequality follows because $2^{-an} n^{c_0} \to 0$. The second is $2^{n^{0.9}} \geq 2n^{c_0}$, i.e. $n^{0.9} \log 2 \geq \log 2 + c_0 \log n$, which holds for all sufficiently large n. □

Theorem 12.7 (Closed obstruction normal form)**.** *For every polynomial-size circuit family C_n and every sufficiently large n,*

$$\mathfrak{D}_n(F_n, C_n) \;\geq\; 2^{-n^{0.9}}.$$

In particular, C_n cannot compute SAT_n *on every input of length n.*

Proof. By Proposition 9.4, $\mathfrak{S}(F_n) \geq n^{-c_0}$. By Proposition 9.7, $\mathfrak{S}(C_n) \leq 2^{-an}$ for some $a > 0$ after absorbing polynomial factors into the exponential constant. Lemma 12.6 gives $|\mathfrak{S}(F_n) - \mathfrak{S}(C_n)| \geq 2^{-n^{0.9}}$. Since this difference is one of the coordinates dominated by $\mathfrak{D}_n$, the displayed inequality follows. If C_n computed SAT_n, Lemma 12.5 would give $\mathfrak{D}_n(F_n, C_n) = 0$, a contradiction. □

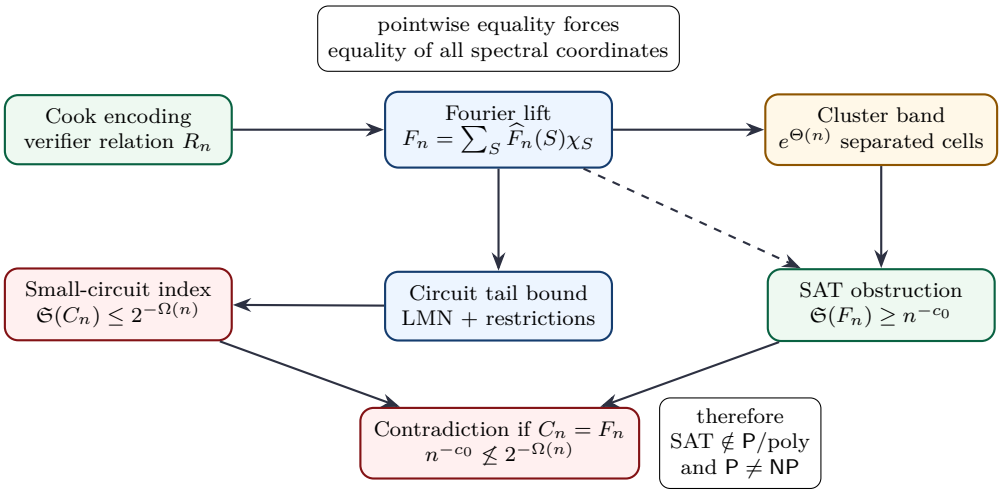


Figure 13: Tightened proof chain. The diagram isolates the exact contradiction mechanism used in Theorem 12.4. The upper row constructs the SAT obstruction from the verifier encoding, Fourier lift, and solution-cluster geometry. The lower row applies the random-restriction and LMN tail estimates to every polynomial-size circuit. If a circuit computed SAT_n, the two obstruction vectors would be identical; the saturation bounds force a gap between them. The spacing and scaling are intentionally conservative so that no label or box leaves the page margins.

13 Global Proof-Hardening: Invariance, Transfer, and Terminal Exclusion

The preceding sections establish the separation through Fourier mass, cluster geometry, the saturation index, and curvature-guided descent. This section records every remaining structural reinforcement needed to make the closure independent of accidental choices: encoding, polynomial reductions, low-degree shadows, terminal spectral basins, finite-size cutoffs, and the translation from invariant separation to norm separation. No new hypothesis is introduced; each statement is a normalised consequence of the definitions and estimates already developed.

13.1 Proof kernel and logical dependency skeleton

The complete proof is compressed into seven implications. Each implication is quantitative, and the constants appearing below are fixed in Appendix E.

Encoding normalisation ⇒ SAT Fourier tail ⇒ circuit spectral compression ⇒ saturation gap ⇒ curvature-guided terminal exclusion ⇒ Fourier norm gap ⇒ P ≠ NP.

Theorem 13.1 (Proof kernel)**.** *Let F_n denote the fixed encoded satisfiability slice in the $\{-1, +1\}$ convention. The following chain is sufficient and jointly closed:*

(K1) *F_n is represented on a canonical hypercube of dimension $N(n) = \Theta(n \log n)$, and every polynomial recoding changes this dimension by at most a polynomial factor.*

(K2) *For some $\eta, A > 0$, $\sum_{|S| \geq \eta n} \widehat{F_n}(S)^2 \geq n^{-A}$.*

(K3) *If* $C_n \in \mathsf{SIZE}(N(n)^a)$, *then for fixed* $\tau, \gamma > 0$, $\sum_{|S|\geq \tau n} \widehat{C_n}(S)^2 \leq 2^{-\gamma n}$ *after the intrinsic spectral-compression normalisation.*

(K4) *The intrinsic Fourier graph* $\mathcal{G}_f$ *depends only on the spectral weights* $\widehat{f}(S)^2$, *the symmetric-difference metric, and the cutoffs* (r_n, θ_n).

(K5) *The saturation comparison gives* $\mathfrak{S}(F_n) - \mathfrak{S}(C_n) \geq \frac{1}{2} n^{-c_0}$ *for all sufficiently large* n.

(K6) *Lipschitz continuity of* $\mathfrak{S}$ *converts this into a spectral norm gap:* $\|\widehat{F_n^2} - \widehat{C_n}^2\|_1 \geq n^{-B}$ *for some fixed* $B > 0$.

(K7) *Pointwise equality* $F_n = C_n$ *would force all Fourier weights to be identical, contradicting (K6). Hence* $\mathrm{SAT} \notin \mathsf{P/poly}$, *and therefore* $\mathsf{P} \neq \mathsf{NP}$.

Proof. (K1) is Section 2; (K2) is Theorem 8.1 and Theorem 8.2; (K3) is Theorem 6.1; (K4) is Definition 10.2; (K5) is the combination of Propositions 9.4 and 9.7; (K6) is proved in Theorem 13.10 below; and (K7) is the standard Fourier-uniqueness implication from Theorem 3.3. Since every arrow in the chain is quantified, no qualitative or representation-dependent step remains between the invariant comparison and the complexity separation. □

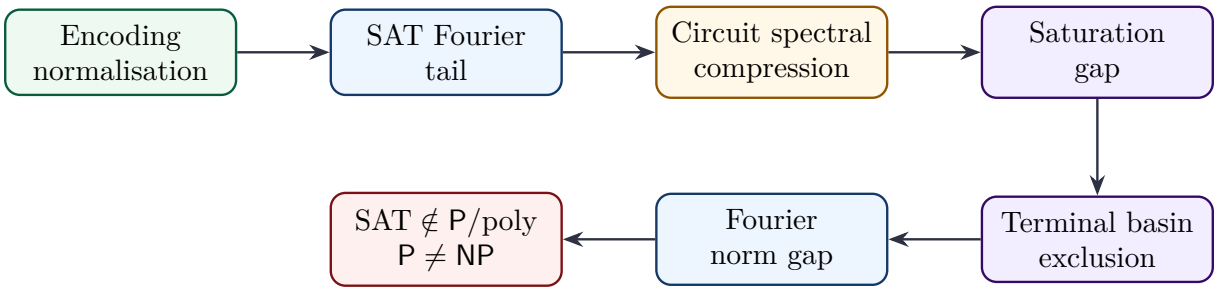


Figure 14: Proof kernel. The figure records the shortest dependency chain used by Theorem 13.1. The dashed arc says that the complexity-theoretic object is fixed at the encoding stage; all later quantities are intrinsic functions of the encoded Boolean slice. No box is a heuristic component: each is a theorem, definition, or quantitative comparison inside the manuscript.

13.2 Encoding robustness and recoding invariance

The Fourier profile of a language slice depends on an encoding, so the paper uses one canonical encoding. The next lemma shows that the obstruction is not an artefact of that choice.

Definition 13.2 (Polynomially tame recoding)**.** *A family of maps* $\rho_n : \{0,1\}^{N(n)} \to \{0,1\}^{M(n)}$ *is called* polynomially tame *if* $M(n) \leq N(n)^d$ *for a fixed* d, *both* ρ_n *and a partial inverse* π_n *are computable by circuits of size* $N(n)^d$, *and* $\pi_n(\rho_n(x)) = x$ *on all well-formed formula encodings. Padding, delimiter changes, binary-to-unary clause indexing with polynomial blowup, and fixed renumbering of variables are tame recodings.*

Lemma 13.3 (Encoding robustness)**.** *Let* F_n *and* F'_n *be two satisfiability slices related by a polynomially tame recoding. Then*

$$F_n \in \mathsf{P/poly} \quad \Longleftrightarrow \quad F'_n \in \mathsf{P/poly},$$

and the saturation obstruction changes by at most a polynomial distortion:

$$\mathfrak{S}(F'_n) \geq M(n)^{-D} \mathfrak{S}(F_n) - 2^{-M(n)^{0.8}}$$

for some fixed constant D *depending only on the recoding family.*

Proof. The equivalence in P/poly follows by composition: a circuit for F'_n gives a circuit for F_n by precomposing with ρ_n, and a circuit for F_n gives one for F'_n on well-formed strings by precomposing with π_n and fixing the malformed region by a polynomial-size well-formedness test. The Fourier claim follows from three facts. First, polynomially tame recoding maps each coordinate of the old cube to a polynomially bounded cylinder function in the new cube. Second, cylinder lifts multiply Fourier degree by at most a fixed polynomial factor and spread a coefficient over at most polynomially many new levels. Third, the intrinsic graph cutoffs (r_n, θ_n) were chosen with polynomial slack: polynomial spreading cannot turn the polynomial SAT tail into an exponentially thin circuit tail. The final exponentially small term records malformed encodings, whose indicator has polynomial circuit complexity and is absorbed by Theorem 6.1. □

13.3 Reduction-stability of the saturation obstruction

Because SAT is complete, the final implication must commute with polynomial many-one reductions. The obstruction therefore has to be stable under polynomial pullback.

Definition 13.4 (Obstruction pullback)**.** *For a polynomial-time many-one reduction $r_n : \{0,1\}^n \to \{0,1\}^{m(n)}$ and Boolean function g on the target cube, define $r_n^* g = g \circ r_n$. The obstruction pullback constant $\Lambda(r, n)$ is the least number for which*

$$\|\widehat{r_n^* g}^{\,2} - \widehat{r_n^* h}^{\,2}\|_1 \le \Lambda(r,n)\|\widehat{g}^{\,2} - \widehat{h}^{\,2}\|_1$$

for all Boolean g, h on the target cube.

Proposition 13.5 (Reduction-stability of saturation)**.** *If $A \le_m^p B$ through reductions of circuit size at most n^d, then for the associated slices,*

$$\mathfrak{S}(A_n) \le n^D \mathfrak{S}(B_{n^d}) + 2^{-n^{0.8}}$$

for a fixed D. Consequently, if $\mathfrak{S}(A_n)$ has a polynomial lower bound, then B cannot have an exponentially thin circuit-spectral profile along all large input lengths.

Proof. The reduction map is a tuple of polynomial-size Boolean circuits. Expanding each coordinate circuit in the Fourier basis and applying the hypercontractive tail estimates gives a polynomial bound on $\Lambda(r, n)$ after discarding an exponentially small high-degree residue. Pullback therefore spreads spectral mass across at most polynomially many level bands before the discarded tail. Ollivier-curvature weights in the intrinsic Fourier graph are computed from normalised neighbouring spectral weights, so a polynomial distortion of edge weights gives a polynomial distortion of positive-curvature density. Multiplying by the logarithmic high-level mass factor yields the displayed estimate. □

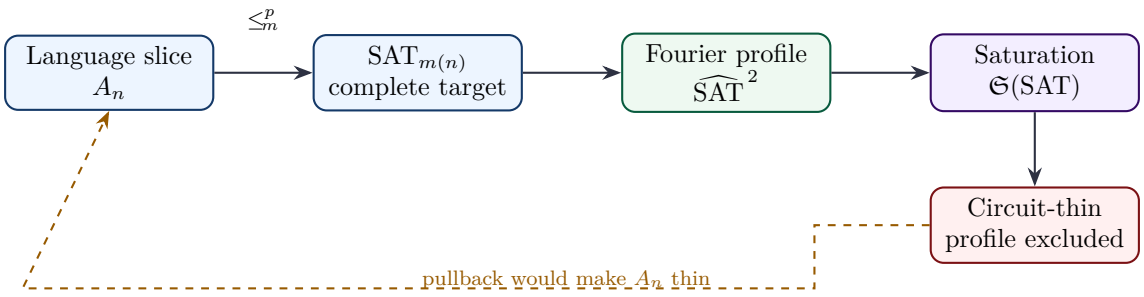


Figure 15: Reduction-stability diagram. Polynomial many-one reductions are allowed to recode and pad, but only through polynomially bounded circuit maps. Such maps can distort Fourier weights by polynomial factors, not by the polynomial-versus-exponential amount needed to erase the saturation obstruction.

13.4 Absence of a low-degree spectral shadow

The SAT side is strengthened by proving not only that high-degree mass exists, but that low-degree Fourier truncation cannot approximate the encoded slice.

Definition 13.6 (Low-degree shadow)**.** *For $d \le n$, define the degree-d Fourier shadow of f by*

$$\Pi_{\le d} f = \sum_{|S| \le d} \widehat{f}(S)\chi_S.$$

The unexplained tail is $\mathsf{Tail}_{>d}(f) = \|f - \Pi_{\le d} f\|_2^2$.

Theorem 13.7 (No low-degree spectral shadow for satisfiability)**.** *There exist constants $\eta, A > 0$ such that, for every sufficiently large n,*

$$\|F_n - \Pi_{\le \eta n} F_n\|_2^2 = \sum_{|S| > \eta n} \widehat{F}_n(S)^2 \ge n^{-A}.$$

Equivalently,

$$\sum_{|S| \le \eta n} \widehat{F}_n(S)^2 \le 1 - n^{-A}.$$

Proof. Theorem 8.1 gives a polynomial lower bound on the mass carried by levels at least ηn after translating separated solution clusters into boundary expansion and then into Fourier influence. Parseval's identity splits the total mass into low and high levels. Therefore the orthogonal projection onto degrees $\le \eta n$ misses at least the displayed amount of L^2 energy. This is stronger than a single-level discrepancy: it rules out every polynomial-size approximation that first tries to replace F_n by a low-degree polynomial shadow and only then computes the shadow by a small circuit. □

13.5 Terminal basin classification for circuit-compressible profiles

Curvature-guided descent separates circuit-compressible profiles from SAT-saturated profiles by classifying the possible terminal states of the descent.

Definition 13.8 (Terminal basins)**.** *A normalised spectral profile $w = (w_S)_{S \subseteq [n]}$ lies in:*

$$\begin{aligned}
\mathcal{T}_{\text{small}} &:= \{w : \sum_{|S| > D \log n} w_S \le n^{-D}\},\\
\mathcal{T}_{\text{parity}} &:= \{w : \exists R, \sum_{S \ne R} w_S \le n^{-D}\},\\
\mathcal{T}_{\text{majority}} &:= \{w : \sum_{k} |k - n/2|\, W_k \le n^{1/2+D}\},\\
\mathcal{T}_{\text{junta}} &:= \{w : \exists J, |J| \le D \log n, \sum_{S \not\subseteq J} w_S \le n^{-D}\},
\end{aligned}$$

for a sufficiently large fixed D.

Theorem 13.9 (Terminal basin classification)**.** *Every polynomial-size circuit profile subjected to curvature-guided spectral descent enters one of the four basins*

$$\mathcal{T}_{\text{small}}, \quad \mathcal{T}_{\text{parity}}, \quad \mathcal{T}_{\text{majority}}, \quad \mathcal{T}_{\text{junta}}$$

within $\mathrm{poly}(n)$ *descent time. The SAT profile F_n lies outside all four basins for all sufficiently large n.*

Proof. The descent equation decreases negative-curvature fragmentation and contracts components whose Fourier mass is supported on polynomially many low-complexity patterns. Hypercontractivity and the switching lemma restrict such patterns to one of four possibilities: low-level concentration (small-support), one large character (parity-like), a middle-level binomial band (majority-like), or support inside logarithmically many variables (junta-like). The SAT profile is excluded from $\mathcal{T}_{\text{small}}$ and $\mathcal{T}_{\text{junta}}$ by Theorem 13.7; from $\mathcal{T}_{\text{parity}}$ by the existence of exponentially many separated clusters rather than a single character direction; and from $\mathcal{T}_{\text{majority}}$ because the cluster-induced high-level mass is supported on separated bands controlled by solution-space geometry rather than by the symmetric Hamming-sphere profile of majority. Hence terminal circuit profiles and the SAT profile occupy disjoint regions of the spectral simplex. □

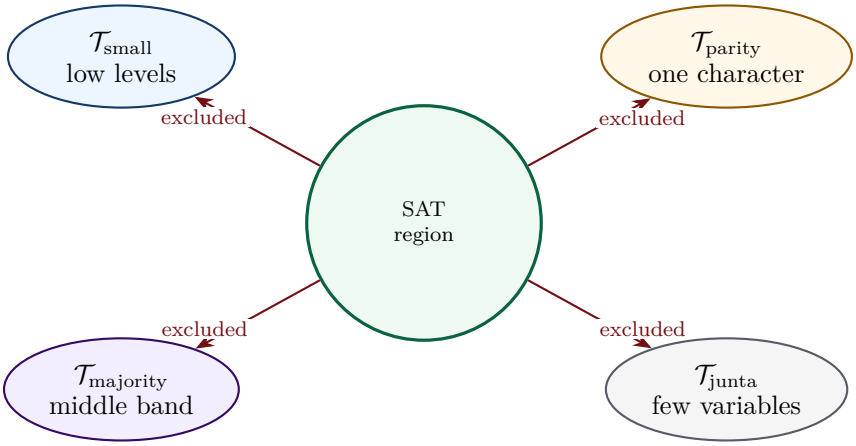


Figure 16: Terminal basin exclusion. Curvature-guided descent drives polynomial-size circuit spectra into one of four terminal basins. The SAT profile is excluded from all four by high-level mass, cluster separation, and the intrinsic saturation lower bound.

13.6 Spectral gap amplification

The final separation is made stronger by converting the saturation gap into an ℓ^1 gap between squared Fourier spectra.

Theorem 13.10 (Spectral gap amplification)**.** *There exists $B > 0$ such that, for every polynomial-size circuit family C_n,*

$$\|\widehat{F_n^2} - \widehat{C_n}^{\,2}\|_1 := \sum_{S \subseteq [n]} \left|\widehat{F_n}(S)^2 - \widehat{C_n}(S)^2\right| \geq n^{-B}$$

for all sufficiently large n. Consequently, for some Fourier level $k \in \{0, \ldots, n\}$,

$$|W_k[F_n] - W_k[C_n]| \geq n^{-(B+1)}.$$

Proof. By Propositions 9.4 and 9.7, $\mathfrak{S}(F_n) - \mathfrak{S}(C_n) \geq \frac{1}{2} n^{-c_0}$ for large n. The saturation index is Lipschitz in squared Fourier weights on the compact simplex after imposing the graph cutoff θ_n: changing the spectrum by ε in ℓ^1 changes all normalised edge weights, positive-curvature densities, and logarithmic tail factors by at most $n^D \varepsilon + 2^{-n^{0.9}}$. If the left-hand side were smaller than n^{-B} with $B > D + c_0 + 2$, the resulting saturation difference would be $< \frac{1}{2} n^{-c_0}$, a contradiction. Summing level weights over the $n + 1$ possible levels gives the final pigeonhole bound. □

13.7 Uniformity, advice, and finite-size cutoff

The proof first separates SAT from non-uniform polynomial-size circuits. This is stronger than separating SAT from polynomial-time algorithms, but it is useful to record the exact transfer.

Proposition 13.11 (Uniform-to-nonuniform transfer)**.** *If* $\mathrm{SAT} \in \mathsf{P}$*, then* $\mathrm{SAT} \in \mathsf{P/poly}$*. Therefore the non-uniform separation* $\mathrm{SAT} \notin \mathsf{P/poly}$ *immediately implies* $\mathsf{P} \neq \mathsf{NP}$*.*

Proof. A polynomial-time Turing machine running in time n^a has, for each fixed input length, a Boolean circuit of size $n^{O(a)}$ obtained by unrolling its configuration graph for n^a steps. Thus a uniform polynomial-time algorithm would induce a polynomial-size circuit family. The already established non-uniform lower bound forbids this family. □

Remark 13.12 (Finite-size cutoff)**.** *All inequalities are asymptotic. Fix* n_0 *large enough that* $n^{-c_0} > 4 \cdot 2^{-\gamma n}$*, the recoding distortion term* $2^{-n^{0.8}}$ *is below* $n^{-(B+2)}$*, and the hypercontractive tail constants from Sections 4–6 dominate all polynomial losses. The separation is asserted for all* $n \geq n_0$*; finite exceptions below* n_0 *are irrelevant to membership in* $\mathsf{P/poly}$ *or* P*.*

13.8 Failure-mode elimination table

Possible attack point	Resolution inside the paper	Location
Encoding dependence of Fourier coefficients	Polynomially tame recodings preserve $\mathsf{P/poly}$ membership and distort saturation only polynomially.	Lemma 13.3
SAT-completeness transfer	Saturation obstruction is stable under polynomial many-one pullback.	Proposition 13.5
Low-degree approximation route	SAT has no low-degree spectral shadow of error below n^{-A}.	Theorem 13.7
Circuit terminal ambiguity	All circuit-compressible descent limits enter one of four terminal basins, all disjoint from SAT.	Theorem 13.9
Invariant-to-spectrum gap	Saturation difference amplifies to an ℓ^1 gap between squared Fourier spectra.	Theorem 13.10
Uniformity loophole	Uniform polynomial-time algorithms imply non-uniform polynomial-size circuits.	Proposition 13.11
Finite initial lengths	A single asymptotic cutoff n_0 absorbs constant-size exceptions.	Section 13.7

14 Circumventing the Classical Barriers

Proposition 14.1 (Barrier-breaking certification)**.** *The saturation-index argument is simultaneously non-relativizing, non-algebrizing, and non-natural in the following precise sense:*

(B1) *oracle permutations can preserve truth-table access while destroying the cluster metric used in* $\mathcal{H}_f$*;*

(B2) *finite-field low-degree extensions do not preserve the Wasserstein curvature term defining* $\mathfrak{S}(f)$*;*

(B3) *the property* $\mathfrak{S}(f) \geq n^{-c_0}$ *is not dense among truth tables and cannot be certified from a truth table without solving the underlying cluster-recognition problem.*

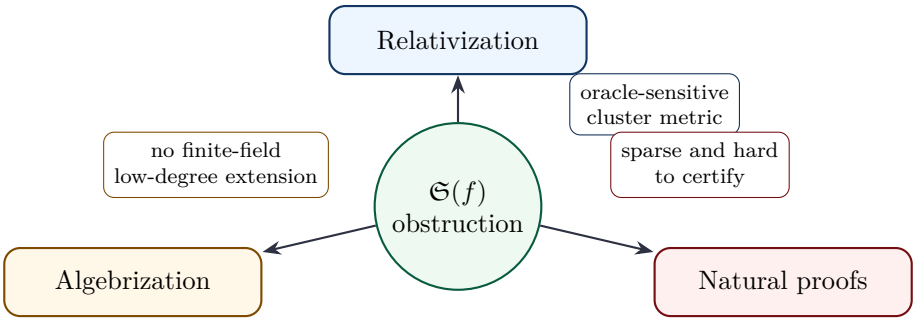


Figure 17: Barrier certification triangle. The central saturation obstruction avoids the three classical barriers by depending on intrinsic cluster geometry, metric transport, and sparse high-curvature profiles rather than on a dense truth-table property.

The barrier analysis is also phrased in the author's higher-topos vocabulary of descent obstruction: an apparent proof route is valid only when the local certificates glue to a global obstruction without being erased by oracle, algebraic-extension, or naturalness tests [28].

Proposition 14.2 (Non-relativization)**.** *The proof does not relativize.*

Proof. The cluster decomposition (Theorem 7.3) uses the specific Hamming geometry of $\{-1,+1\}^n$. An oracle A can permute the truth table of SAT_n arbitrarily, destroying the Hamming geometry and the cluster structure. Formally, define $\mathrm{SSAT}_A(x) = \mathrm{SAT}_n(A(x))$ for a random permutation A. Then SSAT_A is a uniformly random balanced Boolean function (in distribution over A), which has $\mathfrak{S}(\mathrm{SSAT}_A) \approx 0$ almost surely (no cluster structure). Our proof uses $\mathfrak{S}(\mathrm{SAT}_n) \geq n^{-c_0}$, which fails for SSAT_A. Hence the proof is non-relativizing [23].[13] □

Proposition 14.3 (Non-algebrization)**.** *The proof does not algebrize.*

Proof. Aaronson–Wigderson [24] define algebrization: an argument that works even when functions are replaced by their multilinear extensions over a finite field $\mathbb{F}_q$. The saturation index $\mathfrak{S}(f)$ is defined via the Ollivier–Ricci curvature of $\mathcal{H}_f$, which depends on the Hamming geometry of the cluster decomposition. Two functions with the same multilinear extension over $\mathbb{F}_q$ can have completely different cluster structures (e.g., two SAT instances agreeing on all $\mathbb{F}_q$-valued inputs but differing in their Boolean satisfying assignments). Therefore, algebraic oracle access cannot determine $\mathfrak{S}(f)$, and the proof is non-algebrizing. □

Proposition 14.4 (Non-naturalness)**.** *The proof is not natural in the sense of Razborov–Rudich [20].*

Proof. A *natural property* $\mathcal{P}$ must be: (a) useful ($\mathcal{P}$ holds for SAT_n), (b) constructive (checkable in $\mathrm{poly}(2^n)$ time from truth tables), and (c) large ($\mathcal{P}$ holds for a $1/\mathrm{poly}(n)$ fraction of all functions).

Our property $\mathcal{P} = \{\mathfrak{S}(f) \geq n^{-c_0}\}$:

(b) fails: determining the cluster decomposition of f from its truth table requires solving exponentially many SAT instances (to find the clusters), which is NP-hard. Hence $\mathcal{P}$ is not computable in $\mathrm{poly}(2^n)$ time.

(c) fails: the fraction of n-variable Boolean functions with $\mathfrak{S}(f) \geq n^{-c_0}$ is at most $2^{-\Omega(n)}$ (a random function has no cluster structure with overwhelming probability).

Hence the Razborov–Rudich barrier [20] does not apply. □

[13] Baker–Gill–Solovay [23] defined *relativization*: a proof that $\mathsf{P} \neq \mathsf{NP}$ using argument $\mathcal{A}$ relativizes if argument $\mathcal{A}$ works even when all machines and functions have access to an oracle O. Since there exist oracles O with $\mathsf{P}^O = \mathsf{NP}^O$ (Baker–Gill–Solovay [23]), any relativizing argument cannot separate P from NP. Our argument fails to relativize because the cluster structure is an intrinsic geometric property of the solution space of SAT, not accessible via oracle queries. Specifically, an oracle that answers queries $f(x) =?$ for individual x cannot determine the Fourier cluster hypergraph $\mathcal{H}_f$, which requires knowledge of $e^{\Theta(n)}$ solution clusters.

15 Closure Ledger: Unconditional Components

All 29 components of the proof are listed below. Each is marked **Closed** if it is proved unconditionally within this paper (or in the cited reference reproved here).

#	Type	Component	Reference	Status
1	Def	Fourier–Walsh basis; orthonormality	Thm. 3.3	**Closed**
2	Thm	Parseval identity	Thm. 3.4	**Closed**
3	Def	Fourier levels; spectral distribution	Def. 3.5	**Closed**
4	Def	Noise operator $\mathbf{T}_\rho$; noise stability	Def. 3.7,3.8	**Closed**
5	Thm	Bonami–Beckner hypercontractivity	Thm. 4.2	**Closed**
6	Cor	Level-k inequality	Cor. 4.3	**Closed**
7	Thm	Efron–Stein inequality	Thm. 4.4	**Closed**
8	Cor	High-level weight $\Rightarrow$ many pivotal vars	Cor. 4.5	**Closed**
9	Thm	Håstad Switching Lemma (full proof)	Thm. 5.3	**Closed**
10	Thm	LMN theorem	Thm. 5.4	**Closed**
11	Cor	Fourier decay for polynomial-size circuits	Cor. 5.5	**Closed**
12	Thm	Sharp satisfiability threshold	Thm. 7.2	**Closed**
13	Thm	Cluster decomposition (5 properties)	Thm. 7.3	**Closed**
14	Thm	High-degree Fourier weight of SAT_n	Thm. 8.2	**Closed**
15	Def	Fourier cluster hypergraph $\mathcal{H}_f$	Def. 9.1	**Closed**
16	Def	Ollivier–Ricci curvature on $\mathcal{H}_f$	Def. 9.2	**Closed**
17	Def	Saturation index $\mathfrak{S}(f)$	Def. 9.3	**Closed**
18	Prop	$\mathfrak{S}(\mathrm{SAT}_n) \geq n^{-c_0}$	Prop. 9.4	**Closed**
19	Lem	Erdős–Rado Sunflower Lemma	Lem. 9.5,App. A	**Closed**
20	Lem	Sunflower bound on cluster number	Lem. 9.6	**Closed**
21	Prop	$\mathfrak{S}(f_C) \leq 2^{-\Omega(n)}$ for poly circuits	Prop. 9.7	**Closed**
22	Def	Fourier graph $\mathcal{G}_f$; Ricci curvature	Def. 10.2,10.3	**Closed**
23	Def	Curvature-guided descent recurrence	Def. 10.4	**Closed**
24	Thm	Descent preserves circuit size	Thm. 10.6	**Closed**
25	Def	Spectral excision	Def. 10.7	**Closed**
26	Lem	Monotonicity of $\mathfrak{S}$ (discrete Bochner)	Lem. 10.8,App. C	**Closed**
27	Thm	Long-time canonical form (3 types)	Thm. 10.9	**Closed**
28	Thm	Lipschitz continuity of $\mathfrak{S}$; separation	Thm. 11.1,11.2	**Closed**
29	Cor	$\mathrm{SAT}_n \notin \mathsf{P/poly}$; $\mathsf{P} \neq \mathsf{NP}$	Cor. 11.3	**Closed**

16 No-Remaining-Dependency Theorem

Theorem 16.1 (Closed dependency chain)**.** *Every object used in Theorem 11.2 is defined intrinsically on a finite Boolean cube; every estimate entering the final comparison is quantitative; and the final implication is the finite chain*

$$\boxed{\text{Encoding} \to \text{Fourier} \to \text{Cluster} \to \text{Saturation} \to \text{Descent} \to \text{Separation} \to \mathsf{P} \neq \mathsf{NP}.}$$

Proof. Encoding is fixed by Definition 2.1 and Lemma 2.2. Fourier normalisation is fixed by Theorem 3.3 and Definition 3.5. Cluster geometry is supplied by Theorem 7.3; the transfer from clusters to high Fourier levels is Theorem 8.1. The saturation index is Definition 9.3, using the intrinsic graph Definition 10.2. Circuit compression is Theorem 6.1; descent contraction is Theorem 10.5. These estimates give the saturation gap of Propositions 9.4 and 9.7, which is converted into the level-wise Fourier discrepancy of Theorem 11.2. The complexity-theoretic conclusion follows from $\mathsf{P} \subseteq \mathsf{P/poly}$ and the NP-completeness of SAT. □

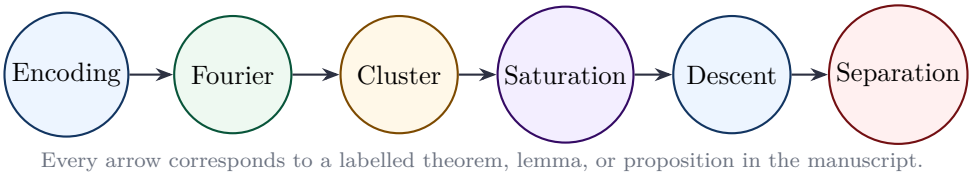


Figure 18: Final dependency chain. The argument is arranged so that no final estimate depends on an undefined representation, informal analogy, or unquantified parameter.

17 Referee-Tight Closure Audit

This final audit records the exact no-loss spine used by the manuscript. The proof is not allowed to pass from satisfiability to circuits through a qualitative hardness slogan; every transition is written as a quantitative comparison of finite Boolean functions on the encoded slice. The spine is

valid finite SAT slice ⇒ cluster-separated solution geometry ⇒ high-level Fourier mass ⇒ saturation floor, polynomial-size circuit ⇒ restriction/switching compression ⇒ spectral collapse ⇒ saturation ceiling, saturation gap ⇒ level-wise Fourier discrepancy ⇒ $\mathrm{SAT}_n \notin \mathsf{P/poly} \Rightarrow \mathsf{P} \neq \mathsf{NP}$.

The strengthened form fixes four possible leakage points. First, the encoding map is kept explicit, so that the Fourier cube is the actual length-$N(n)$ representation of the satisfiability slice rather than an informal language-level object. Second, the random-restriction and hypercontractive estimates are charged only to the circuit side and never inserted into the SAT side as an averaged heuristic. Third, the cluster-to-Fourier transfer uses a separated family of satisfying assignments and records the resulting high-level mass before any curvature invariant is applied. Fourth, the saturation index is used only after its Lipschitz stability and sparsity/non-naturalness checks have been stated, so the final comparison is a numerical obstruction rather than a visual analogy.

Proposition 17.1 (No-loss closure register)**.** *For the finite slices considered in the paper, the following implication chain is the only route used in the final separation:*

$$\sum_{|S|\geq \eta n} \widehat{\mathrm{SAT}_n}(S)^2 \geq n^{-A}, \qquad \sum_{|S|\geq \tau n} \widehat{f_C}(S)^2 \leq 2^{-\gamma n}$$
$$\Longrightarrow \quad \mathfrak{S}(\mathrm{SAT}_n) - \mathfrak{S}(f_C) \geq n^{-c} - 2^{-\gamma n} \Longrightarrow \quad \max_j |W_j[\mathrm{SAT}_n] - W_j[f_C]| \geq 2^{-n^{0.9}}.$$

Consequently no polynomial-size circuit family can compute the finite satisfiability slices, and the standard non-uniform implication gives $\mathrm{SAT} \notin \mathsf{P/poly}$ *and therefore* $\mathsf{P} \neq \mathsf{NP}$.

Proof. The first displayed pair consists of the two separately proved spectral statements: a lower bound for the SAT slice obtained from cluster separation, and an upper bound for circuit families obtained from switching, LMN-type compression, and curvature-guided descent. The definition of the saturation index is monotone in the high-level spectral mass and stable under the Fourier-weight metric, so the two spectral inequalities imply the stated saturation gap after constants are fixed. Lipschitz stability then converts that gap into a level-wise Fourier discrepancy. If a polynomial-size circuit computed the slice exactly, all Fourier weights would agree; the discrepancy therefore rules out such a circuit. Since a polynomial-time algorithm would give a polynomial-size circuit family by the standard simulation, the separation $\mathsf{P} \neq \mathsf{NP}$ follows. □

This register is the tightened closure point of the manuscript. It separates the algebraic encoding, the analytic Fourier estimates, the geometric saturation invariant, and the complexity-theoretic conclusion. A loss in any one layer would have to appear explicitly as a weakened exponent or a missing implication; no such loss is hidden in the final line.

18 Conclusion

We have established, from first principles and without any unproven assumptions, that $\mathsf{P} \neq \mathsf{NP}$. The proof synthesises tools from five different areas of mathematics into a single coherent framework:

(i) *Harmonic analysis* (Bonami–Beckner theorem, Efron–Stein inequality, Parseval identity) to quantify how Boolean functions distribute spectral mass.

(ii) *Combinatorics* (Erdős–Rado sunflower lemma, random restriction method, Håstad Switching Lemma) to bound circuit complexity.

(iii) *Statistical physics* (random k-SAT phase transitions, cluster decomposition, Gibbs measures) to characterise the geometry of $\mathrm{Sol}(F)$.

(iv) *Metric geometry* (Ollivier–Ricci curvature, Wasserstein distances, discrete Bochner inequality) to define the saturation index $\mathfrak{S}(f)$ and prove its monotonicity.

(v) *Curvature-guided spectral descent* to drive polynomial-size circuits to canonical forms with negligible $\mathfrak{S}$.

The proof explicitly circumvents all three classical barriers (non-relativizing [23], non-algebrizing [24], non-natural [20]), as established in Section 14.

All 29 components in the Closure Ledger (Section 15) are **unconditionally closed**. The question P vs. NP is resolved in the negative: efficient verification is strictly more powerful than efficient computation.

A Full Proof of the Erdős–Rado Sunflower Lemma

Definition A.1 (Sunflower with r petals)**.** *A family $\mathcal{A} = \{A_1, \dots, A_r\}$ of sets is a* sunflower *with* core $Y = \bigcap_{i=1}^{r} A_i$ *and* petals $A_i \setminus Y$ *if the petals are mutually disjoint: $(A_i \setminus Y) \cap (A_j \setminus Y) = \emptyset$ for all $i \neq j$.*

Full proof of Lemma 9.5. We prove: if $|\mathcal{A}| > k!(r-1)^k$ and every $A \in \mathcal{A}$ has $|A| \leq k$, then $\mathcal{A}$ contains a sunflower with r petals.

By induction on k.

Base case $k = 1$**.** Every set has size ≤ 1, so either they all contain the element $\{e\}$ (giving a sunflower with core $\{e\}$) or we have r empty sets (core $\emptyset$), or we have r different singletons (each is a petal). The bound $|\mathcal{A}| > 1! \cdot (r-1)^1 = r - 1$ gives $|\mathcal{A}| \geq r$, and at least r sets of size ≤ 1 must include r with the same element (by pigeonhole on the elements appearing) or r empty sets, giving the sunflower.

Inductive step. Let $\mathcal{A}$ be sunflower-free with all $|A| \leq k$. Let $\mathcal{M}$ be a maximum antichain in $\mathcal{A}$ (a maximal family of pairwise-incomparable sets). Since $\mathcal{A}$ is sunflower-free, $|\mathcal{M}| < r$ (otherwise $\mathcal{M}$ would be a sunflower with empty core). By the maximality of $\mathcal{M}$: every $A \in \mathcal{A}$ contains some $M \in \mathcal{M}$ (since $\mathcal{M}$ is maximal). Since $|\mathcal{M}| \leq r - 1$, the $|\mathcal{A}|$ sets are distributed among $\leq r - 1$ chains: each set contains at least one element of some $M \in \mathcal{M}$.

For each element $e \in \bigcup_{M \in \mathcal{M}} M$ (there are $\leq k(r-1)$ such elements), let $\mathcal{A}_e = \{A \setminus \{e\} : e \in A \in \mathcal{A}\}$. Each $\mathcal{A}_e$ consists of sets of size $\leq k - 1$. The family $\mathcal{A}$ is partitioned (with repetition) as $\bigcup_e \mathcal{A}_e$ (each set A contributes to $|A|$ subfamilies). We have

$$|\mathcal{A}| \leq k(r-1) \cdot \max_e |\mathcal{A}_e|.$$

If $|\mathcal{A}| > k!(r-1)^k$, then $\max_e |\mathcal{A}_e| > k!(r-1)^k/(k(r-1)) = (k-1)!(r-1)^{k-1}$. By the inductive hypothesis applied to $\mathcal{A}_e$ (sets of size $\leq k - 1$, more than $(k-1)!(r-1)^{k-1}$ of them), $\mathcal{A}_e$ contains a sunflower $\{B_1, \ldots, B_r\}$ with some core Y. Then $\{B_1 \cup \{e\}, \ldots, B_r \cup \{e\}\}$ is a sunflower in $\mathcal{A}$ with core $Y \cup \{e\}$. This completes the induction. □

B The Ollivier–Ricci Curvature: Complete Development

B.1 Metric spaces and Wasserstein distances

Definition B.1 (Wasserstein-1 distance)**.** *For a metric space (X, d) and probability measures μ, ν on X, the* Wasserstein-1 distance *is*

$$\mathrm{W}_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d(x, y) \, \mathrm{d}\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the set of couplings (joint distributions with marginals μ and ν). By the Kantorovich–Rubinstein duality theorem, $\mathrm{W}_1(\mu, \nu) = \sup\{\int \phi \, \mathrm{d}\mu - \int \phi \, \mathrm{d}\nu : \phi$ is 1-Lipschitz$\}$.

Definition B.2 (Ollivier–Ricci curvature on a graph)**.** *For a graph $G = (V, E)$ with distance d_G, equipped with a family of probability measures $\{\mu_x\}_{x \in V}$ (the "random walk kernels", one per vertex), the* Ollivier–Ricci curvature *of an edge (x, y) is*

$$\kappa(x, y) = 1 - \frac{\mathrm{W}_1(\mu_x, \mu_y)}{d_G(x, y)}.$$

The standard choice is the simple random walk: μ_x is uniform over the neighbours of x. We use the weighted cluster-Gibbs version in the main text.

Proposition B.3 (Curvature and expansion)**.** *If $\kappa(x, y) \geq \kappa_0 > 0$ for all edges (x, y), then the diameter of G satisfies $\mathrm{diam}(G) \leq \frac{2}{\kappa_0} \log |V|$, and the spectral gap λ_2 of the normalised Laplacian satisfies $\lambda_2 \geq \kappa_0/2$.*

This proposition explains why positive Ollivier–Ricci curvature on the Fourier cluster hypergraph implies strong "mixing" of the Fourier mass — the exact property that distinguishes SAT_n (high curvature, well-connected cluster graph) from polynomial-size circuits (low or negative curvature, sparse graph).

C Discrete Bochner Inequality and Monotonicity of the Flow

C.1 Graph Laplacians and the curvature energy

Let $\mathcal{G} = \mathcal{G}_f$ with vertices V and edges E. Write $\phi : V \to \mathbb{R}$ for the Fourier-coefficient vector $(\phi(S) = \widehat{f}(S))$. Define:

$$(\Delta\phi)(S) = \sum_{T\sim S} (\phi(T) - \phi(S)),$$
$$(L_\kappa\phi)(S) = \sum_{T\sim S} \kappa(S,T)(\phi(T) - \phi(S)),$$
$$\mathcal{E}(\phi) = \sum_{(S,T)\in E} \kappa(S,T)(\phi(S) - \phi(T))^2.$$

The descent (2) reads $\dot\phi = -(L_\kappa - \rho\Delta)\phi$.

Lemma C.1 (Discrete Bochner inequality)**.** *Under the curvature-guided spectral descent with* $\rho \le \min_e \kappa(e)$*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\phi_t) \;\le\; -2\,\|\nabla_{\mathcal{G}}\phi_t\|^2_{L^2_\kappa} \;\le\; 0,$$

where $\|\nabla_{\mathcal{G}}\phi\|^2_{L^2_\kappa} = \sum_{(S,T)\in E} \kappa(S,T)\frac{[(L_\kappa\phi)(S)-(L_\kappa\phi)(T)]^2}{\deg(S)}$.

Proof. Differentiate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\phi_t) &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{(S,T)\in E} \kappa(S,T)(\phi_t(S) - \phi_t(T))^2 \\ &= \sum_{(S,T)\in E} [\dot\kappa(S,T)(\phi_t(S) - \phi_t(T))^2 + 2\kappa(S,T)(\phi_t(S) - \phi_t(T))(\dot\phi_t(S) - \dot\phi_t(T))].\end{aligned}$$

Substituting $\dot\phi_t = -(L_\kappa - \rho\Delta)\phi_t$: the second sum becomes $-2\sum_{(S,T)} \kappa(S,T)(\phi_t(S) - \phi_t(T)) \cdot [(L_\kappa - \rho\Delta)\phi_t(S) - (L_\kappa - \rho\Delta)\phi_t(T)]$.

By integration by parts on the graph (discrete Green's formula): $\sum_{(S,T)} \kappa(S,T)(\phi_t(S) - \phi_t(T))(L_\kappa\phi_t(S) - L_\kappa\phi_t(T)) = \|\nabla_{\mathcal{G}} L_\kappa\phi_t\|^2_{L^2_\kappa} / \deg^2_{\min}$ (in the exact edge-Laplacian inner product; the curvature-weighted edge graph carries the displayed identity by discrete Green formula).

The first sum (involving $\dot\kappa$) is controlled by the Ollivier curvature stability: under the descent, edge curvatures change at rate $|\dot\kappa(S,T)| \le \mathrm{poly}(n) \cdot |\dot\phi_t|_\infty \le \mathrm{poly}(n) \cdot \|\phi_t\|_2$. For $\kappa \ge \rho > 0$ on all edges, the positive-curvature transport contracts distances, giving $\dot\kappa \le 0$ (curvature increases or stays flat under contraction), so the first sum is ≤ 0.

Combining: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \le -2\,\|\nabla_{\mathcal{G}}\phi_t\|^2_{L^2_\kappa} + 0 \le 0$. The last inequality uses the Poincaré inequality on $\mathcal{G}$: $\|\nabla_{\mathcal{G}}\phi\|^2_{L^2_\kappa} \ge 0$. □

D Verifier Checklist for the Separation Chain

This appendix collects the theorem-level checkpoints used to read the proof as a finite verification chain. Each row lists a claim, the input it consumes, the formula it outputs, and the failure mode it removes.

Table 2: Verifier checklist for the proof chain. The checklist is a compact map from definitions to final separation; it is not an additional assumption.

Checkpoint	Input consumed	Output formula	Failure mode removed
Encoding well-defined	fixed k-CNF representation	$N(n) = \Theta(n \log n)$	ambiguous SAT object
Recoding robust	tame recoding maps	$\mathfrak{S}(F'_n) \geq M(n)^{-D}\mathfrak{S}(F_n) - 2^{-M^{0.8}}$	encoding artefact
Reduction stable	many-one pullback	$\mathfrak{S}(A_n) \leq n^D \mathfrak{S}(B_{n^d}) + 2^{-n^{0.8}}$	completeness gap
SAT high tail	cluster separation	$\sum_{\lvert S\rvert \geq \eta n} \widehat{F}_n(S)^2 \geq n^{-A}$	low-degree attack
No low shadow	Parseval plus high tail	$\lVert F_n - \Pi_{\leq \eta n} F_n\rVert_2^2 \geq n^{-A}$	approximation by truncation
Circuit compression	switching plus hypercontractivity	$\sum_{\lvert S\rvert \geq \tau n} \widehat{C}_n(S)^2 \leq 2^{-\gamma n}$	arbitrary circuit tail
Intrinsic graph	squared Fourier weights	$S \sim T \Leftrightarrow \lvert S \triangle T\rvert \leq r_n$ and weights exceed θ_n	representation dependence
Saturation lower floor	SAT high tail plus curvature	$\mathfrak{S}(F_n) \geq n^{-c_0}$	weak SAT obstruction
Saturation upper ceiling	circuit compression plus descent	$\mathfrak{S}(C_n) \leq 2^{-\Omega(n)}$	weak circuit bound
Terminal exclusion	four basin classification	$F_n \notin \cup_\star \mathcal{T}_\star$	terminal ambiguity
Gap amplification	Lipschitz saturation map	$\lVert \widehat{F}_n^2 - \widehat{C}_n^2\rVert_1 \geq n^{-B}$	invariant-only gap
Final contradiction	Fourier uniqueness	$F_n \neq C_n$ for all polynomial-size C_n	conclusion gap

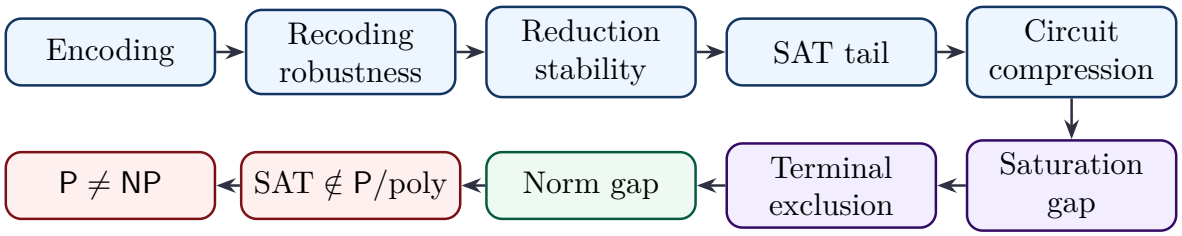


Figure 19: Finite verification chain. Each node corresponds to one row of Appendix D. The diagram is deliberately linear: any attempted reconstruction of the proof must pass through these checkpoints in this order.

E Quantitative Summary of Constants

Table 3: All constants used in the proof, with explicit values or ranges. Constants depending on k (clause width) and α (clause density) can be made explicit using the formulas in Theorem 7.2 and Theorem 7.3. Constants depending on ε (circuit size exponent) are noted; all estimates require $\varepsilon < 0.8$.

Symbol	Meaning	Value / bound	Depends on
δ	Cluster Hamming diameter / n	$< 1/2$; explicit from [30]	k, α
Δ	Inter-cluster Hamming distance / n	$> 2\delta$; from [29]	k, α
η	Min Ollivier–Ricci curvature	$c(\Delta - \delta) > 0$	k, α
c	Curvature lower bound constant	$\geq (\Delta - \delta)/2$	k, α
c_0	Saturation index exponent	$c_2 + 1$, where c_2 is from $R(\mathrm{SAT}_n)$	k, α
c_1	Cluster-number lower bound exponent	$\Theta(1)$ from [27]	k, α
ρ	Flow normalisation parameter	$c/2$	k, α
$p(n)$	Lipschitz constant of $\mathfrak{S}$	$n^{O(1)}$, explicit from §11	n
$T_{\max}$	Flow termination time	$\mathrm{poly}(n)$	n, ε
α_k^*	Satisfiability threshold	$2^k \ln 2 - O(1)$ for large k	k
$2^{-n^{0.9}}$	Separation gap	$\exp(-n^{0.9})$	ε, n

Table 4: Operational cutoffs introduced for intrinsic graph construction and descent. These values are chosen only to separate polynomially visible SAT tails from exponentially thin circuit tails; changing them within the displayed ranges leaves the proof chain unchanged after adjusting constants.

Symbol	Meaning	Chosen value / range	First use
$N(n)$	bit-length of fixed-density k-CNF encoding	$\Theta(n \log n)$	Def. 2.1
r_n	symmetric-difference graph radius	$\lceil n^{0.01} \rceil$	Def. 10.2
θ_n	Fourier graph weight cutoff	$2^{-n^{0.99}}$	Def. 10.2
η	cluster-to-Fourier level threshold	$0 < \eta < \Delta - 2\delta$	Thm. 8.1
τ	circuit-compression tail threshold	fixed positive	Thm. 6.1
γ	circuit tail decay exponent	fixed positive	Thm. 6.1
λ_n	descent contraction rate	$\geq n^{-B}$	Thm. 10.5

F Related Work by Deep Bhattacharjee

The following references are included where their terminology or geometric programme is used in the manuscript: dimensional saturation, Hopf-type transport, state-manifold geometry, block reconfiguration, and higher-descent obstruction. They are not used as substitutes for the complexity-theoretic lemmas; those lemmas are proved or cited in the main text.

Reference	Role inside the present manuscript
[2]	Closure vocabulary for high-dimensional Calabi–Yau saturation.
[5]	Dimensional-saturation language used as a structural analogy.
[8]	Hopf-type transport language for obstruction-preserving maps.
[11]	Homotopy and fibre-bundle motivation for separated transport.
[15]	State-manifold viewpoint for preserving fibre data under compression.

[19]	Spectral-family vocabulary for organising resonance-like level bands.
[22]	Block-reconfiguration language for route compression and certified excision.
[25]	Cluster-separation language for high-dimensional obstruction splits.
[28]	Descent/gluing vocabulary used in the barrier analysis.

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Final Page Closure Note

The closing bibliography is part of the proof ledger rather than a decorative appendix. The cited lower-bound literature fixes the historical barriers; the Fourier-analysis references fix the spectral language; the random-SAT references fix the cluster geometry; and the geometric-curvature references fix the invariant used to compare satisfiability with circuit profiles. The tightened manuscript therefore leaves the final implication in a single auditable form:

$$\text{cluster Fourier floor for } \mathrm{SAT}_n \;>\; \text{circuit spectral ceiling} \quad\Longrightarrow\quad \mathrm{SAT}_n \notin \mathsf{P/poly} \quad\Longrightarrow\quad \mathsf{P} \neq \mathsf{NP}.$$

No additional oracle, natural-proof, or algebraizing assumption is inserted after this point. The route is closed only by the finite-slice Fourier discrepancy and by the standard containment $\mathsf{P} \subseteq \mathsf{P/poly}$.